Mathematics in the New Zealand Curriculum
Second Tier

Strand: Number and Algebra Thread: Patterns and Relationships Level: Five

Achievement Objectives:

- Generalise the properties of operations with fractional numbers and integers.
- Relate tables, graphs and equations to linear and simple quadratic relationships found in number and spatial patterns.

Exemplars of student performance:
Exemplar One:
Sonny (student) solves equations involving fractions with unknowns on both sides of the equals sign. He is able to solve the equations relationally. This means that he applies invariant number properties rather than calculating the total value for each side of the equation. Sonny understands the equals sign as a statement of relational balance that is preserved under certain transformations performed on the expressions in the equation.
To solve:
a. $^{a}/_{12} = {}^{b}/_{36}$, Express the value of b using the variable *a,* and the value of *a* using the variable *b*. Since 36 = 3 x 12 Sonny knows that for any value of *a*, *b* will be three times *a*, and *a* will be one-third *of b*. He writes $b = 3a$ and $a = {}^{1}/_{3}\, b$ or $a = b/_{3}$.

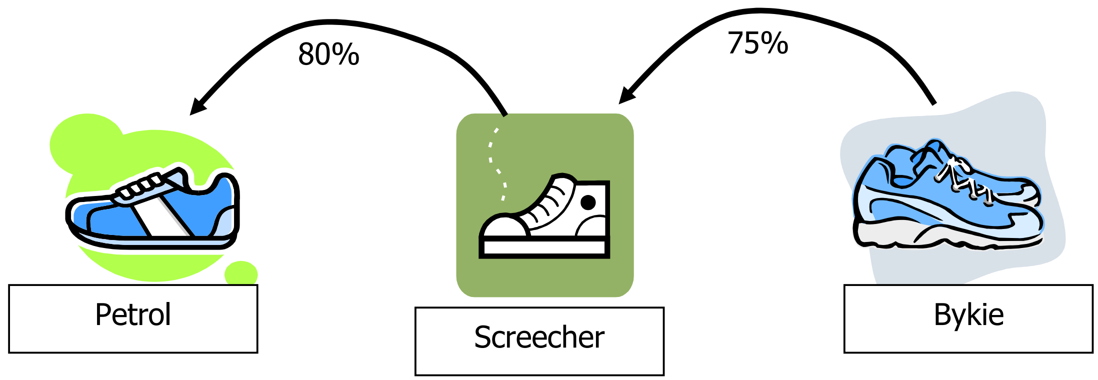
b. $^{2}/_{3} < {}^{18}/_{n}$, For what values of *n* will this statement be true. Sonny recognizes that $^{2}/_{3}$ and $^{18}/_{27}$ are equivalent fractions and that decreasing the denominator while keeping the numerator constant increases the value of a fraction. So he concludes that the statement will be true for values of *n* that are less than 27 but greater than zero, i.e. $0 < n < 27$.

C $^{q}/_{4} + {}^{q}/_{3} = {}^{r}/_{24}$, Express *q* in terms of *r* and vice versa. Sonny recognizes that, to form equivalent fractions with a denominator of 24, $6q$ and $8q$ will be the numerators ($^{q}/_{4} = {}^{6q}/_{24}$ and $^{q}/_{3} = {}^{8q}/_{24}$) . Therefore $r = 14q$.

d. $1 \div {}^{4}/_{u} = w$, Express *w* in terms of *u*. Sonny knows the equality $1 \div {}^{4}/_{u} = 1 \times {}^{u}/_{4}$ (using reciprocals) . So $w = {}^{u}/_{4}$.

Sonny's responses show achievement at level five because he thinks relationally about equations involving fractions and operations on fractions. This shows that he understands the key number properties involved in these operations, including the existence of common factors in equivalent fractions, the need for common denominators in addition/subtraction of fractions, and division by a fraction as equivalent to multiplication by the reciprocal. He is able to treat the equals sign as a statement of both balance (equality) and transitivity (same transformation to both sides).

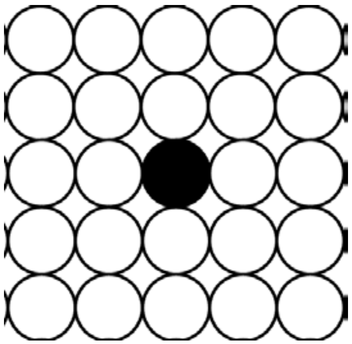
Exemplar Two:
Rebecca (student) has the following problem:
Screecher shoes are 80 percent of the price of Petrol shoes. Bykie shoes are 75% of the price of Screechers.
What percentage of the price of a pair of Petrol shoes is a pair of Bykies?



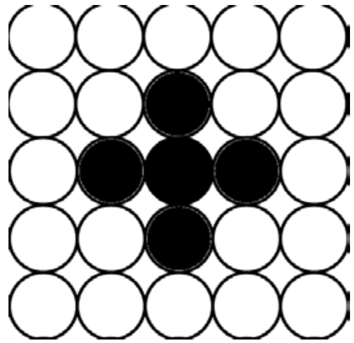
Rebecca recognizes that 80% = $^{4}/_{5}$ and 75% = ¾. She models the problem as $^{4}/_{5}$ x ¾ = $^{12}/_{20}$. She rcognises that $^{12}/_{20}$ = $^{60}/_{100}$ = 60% and that this relationship applies to all the possible values for Bykie and Petrol shoes. Rebecca concludes that Bykie shoes are 60% of the price of Petrol shoes. She checks her conclusion numerically by giving Petrol shoes a hypothetical value of $100.

Rebecca's response exemplifies level five because she accepts lack of closure, that the price of Petrol shoes can be treated as a variable with no attached value. She operates on this variable using her knowledge of fraction-percentage conversions and of multiplication of fractions. She validates her algebraic answer by substituting values for the shoes.

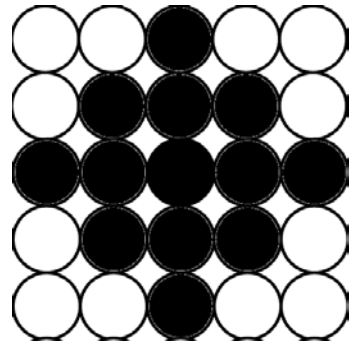
Exemplar Three:
Lambert, Jee Hyun and Kayla work out the number of rotten apples that will be in the tray after ten days if the growth pattern continues as shown.



1 day…



2 days…



3 days…

a. Lambert makes a table to show the relation between the number of days and the number of rotten apples. He looks at the second order differences and realizes that they are increasing by four each time:

Days	1	2	3	4	…	10
Rotten Apples	1	5	13	25	…	?

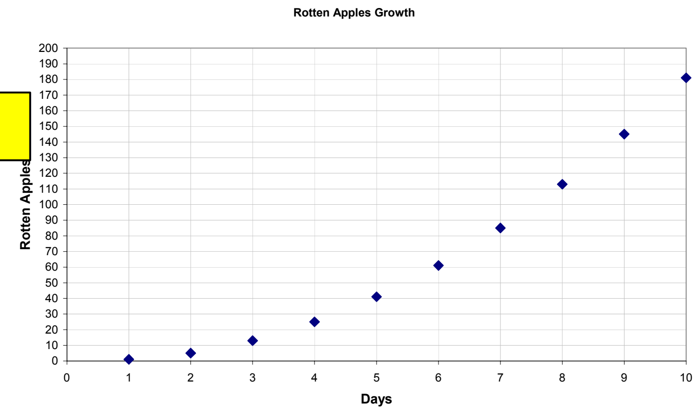
+ 4   + 8   + 12

Lambert uses a spreadsheet to model further ordered pairs for this relation. He uses the graphing capability of the spreadsheet to produce a graph of the relation recognizing that an increasing second order difference will result in a curve with increasing slope called a parabola. He associates this with a relation involving powers of two (a quadratic).

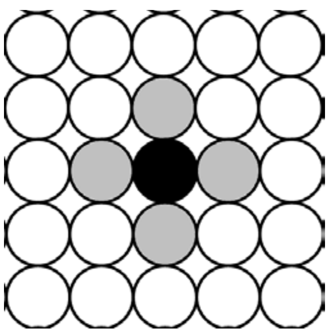
	A	B	C
1	Days	Rotten Apples	Differences
2	1	1	
3	2	5	4
4	3	13	8
5	4	25	12
6	5	41	16
7	6	61	20
8	7	85	24
9	8	113	28
10	9	145	32
11	10	181	36

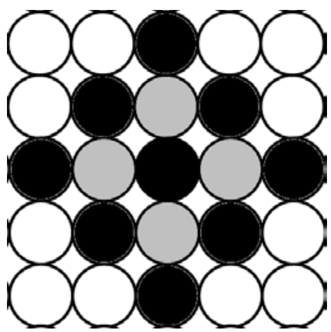
=B3+C4

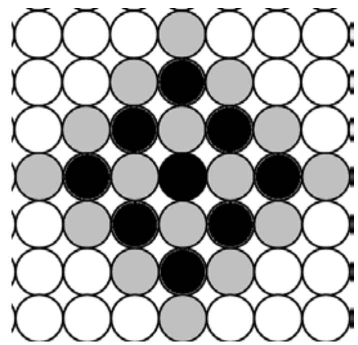
=C3+4



b. Jee Hyun attends to the visual pattern in the total number of apples for a given number of days.







2 days $2^2+1^2 = 5$ 3 days $3^2+2^2 = 13$ 4 days $4^2+3^2 = 25$

She concludes that for ten days the total number of rotten apples will be $10^2 + 9^2 = 100 + 81 = 181$ and that the general rule can be written $a = d^2 + (d-1)^2$, where a represents the number of rotten apples and d represents the number of days.

c. Kayla also notices a pattern in the table of numbers

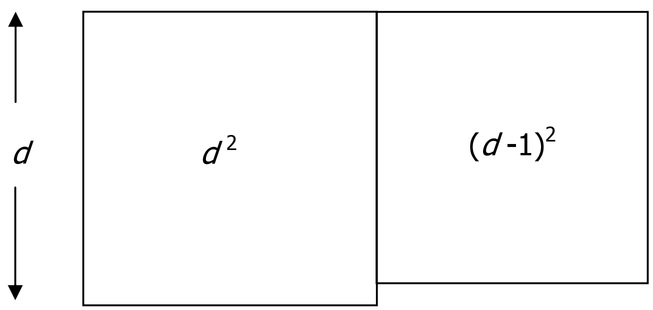
Days	1	2	3	4	...	10
Rotten Apples	1	5	13	25	...	?

$2 \times 2^2 - (2 \times 2) + 1$

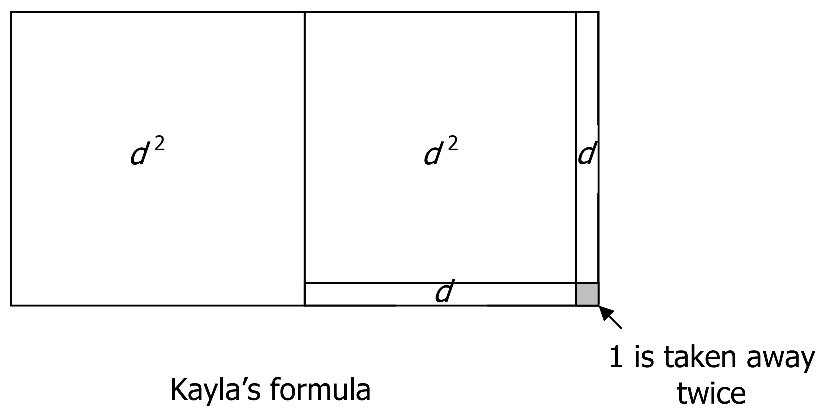
$2 \times 3^2 - (2 \times 3) + 1$

$2 \times 4^2 - (2 \times 4) + 1$

Kayla writes a formula for what she has noticed using d as the variable for number of days, $2d^2 - 2d + 1$. She compares her formula with Jee Hyun's knowing that they both result in the same values for any number of rotten apples (a). Kayla checks her formula geometrically:



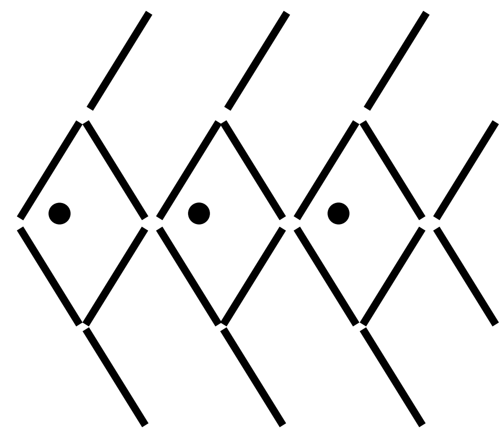
Jee Hyun's formula



Kayla's formula

She recognizes that the formulas are structurally equivalent.

Lambert, Jee Hyun and Kayla all show level five achievement as they use strategies to find missing values in a simple quadratic relation. Lambert uses the power of the spreadsheet environment to model the recursive increase in the values for the number of rotten apples. He also recognises the graph of the relation as a parabolic curve and connects this with a quadratic formula. Jee Hyun recognizes that the pattern can be partitioned geometrically into two square numbers [n^2 and $(n-1)^2$] and generates a formula from this observation. Rebecca looks for patterns formed by the numbers in a table of the relation and seeks to validate her formula, firstly by substituting in numeric values, then linking it geometrically with Jee Hyun's formula.



Exemplar Four:

Jessica solves the following problem:

If this pattern continued in the same way and you used 188 matchsticks to build it, how many fish would there be?

Jessica builds a table of values to organise her counting data:

Number of fish	1	2	3	4	...	?
Number of matches	8	14	20	26	...	188

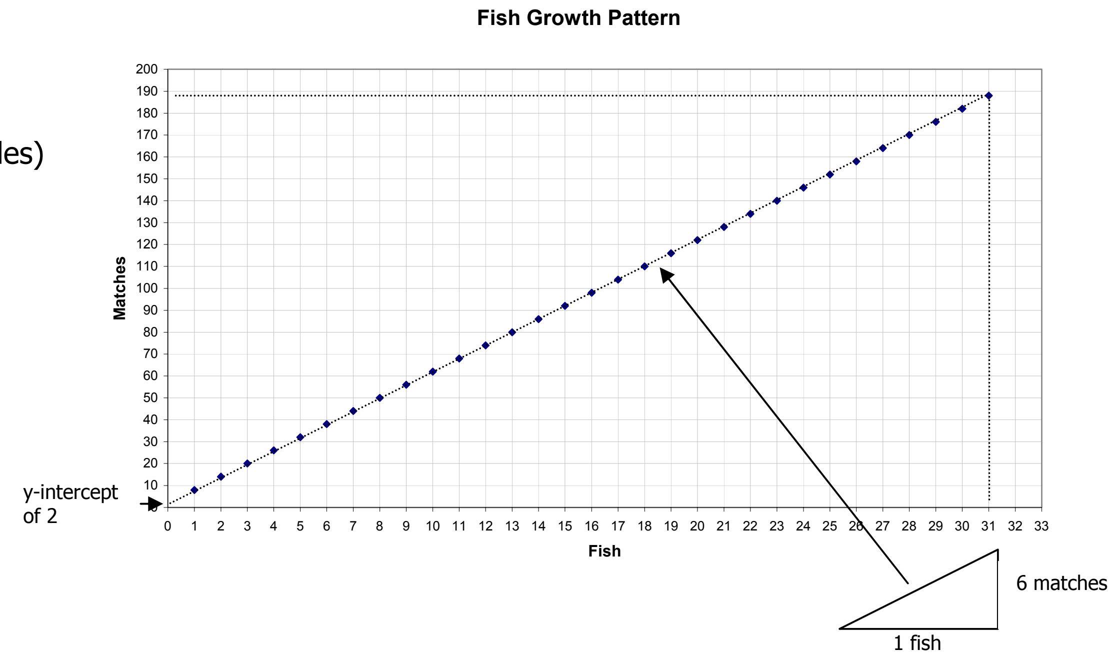
She recognizes that the relation is linear because the number of matches increases by a constant, six, each time another fish is added. Jessica writes an equation for the relation, $m = 6f + 2$, where m represents the number of matches and f, the number of fish. She predicts that a graph of this relation will produce points along a line with slope of six (the constant rate of increase) and a y-intercept of two (corresponding to the added constant in the equation). She knows that this linear graph could be used

to solve the problem.

Jessica uses her linear equation to solve the problem.
She writes the following steps:

$188 = 6f + 2$ (setting m equal to 188)
$\rightarrow 186 = 6f$ (subtracting two from both sides)
$\rightarrow 31 = f$ (Dividing both sides by six)

She checks her calculation by substituting $f = 31$ into her formula $m = 6f + 2$.
The result, 188, confirms her answer.



Jessica's achievement exemplifies level five because she is able to connect several representations of a linear relation to solve the problem. She understands that for a given linear equation, $y = mx + c$, m gives the slope of the line through the graph of the set of ordered pairs (x,y), and c gives the intercept of that line with the y-axis. She is able to apply operations to both sides of a linear equation to solve for one variable and substitutes numerical values back into the equations to check her answers.

Exemplar Five:

Andre investigates this pattern of equations:

$1 \times 1 = 1 + 1$	$1 \times 4 + 2 = {}^{-}2 \times {}^{-}3$	$\frac{1}{1} + \frac{1}{2} = \frac{3}{2}$
$2 \times 3 = 4 + 2$	$2 \times 5 + 2 = {}^{-}3 \times {}^{-}4$	$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$
$3 \times 4 = 9 + 3$	$3 \times 6 + 2 = {}^{-}4 \times {}^{-}5$	$\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

He records other equations that he believes fit the same pattern, by investigating the equations relationally (patterns across) as well as investigating the set of equations recursively (patterns down);

e.g. $76 \times 77 = 76^2 + 76$ $\quad 100 \times 103 + 2 = {}^{-}101 \times {}^{-}102$ $\quad \frac{1}{30} + \frac{1}{31} = \frac{61}{930}$.

Andre uses the letter n to represent a chosen number in each equation and writes a general rule for the equation set.

$n(n + 1) = n^2 + n$ $\quad n(n + 3) + 2 = {}^{-}(n+1){}^{-}(n+2)$ $\quad \frac{1}{n} + \frac{1}{n+1} = \frac{2n+1}{n(n+1)}$

By connecting with the distributive property for multiplication, and the multiplication of integers, he is able to explain how the expressions can be manipulated to show equality.

Equation Set One: As $3 \times (3 + 1) = 3 \times 3 + 3 \times 1$ so;

$$\begin{aligned} n(n + 1) &= n \times n + n \times 1 \\ &= n^2 + n \end{aligned}$$

Equation Set Two: As $(7 + 1) \times (7 + 2) = 7 \times 7 + 7 \times 2 + 1 \times 7 + 1 \times 2$

$= 10 \times 7 + 2$ so;

$$\begin{aligned} {}^{-}(n+1){}^{-}(n+2) &= (n+1)(n+2) \\ &= n^2 + 2n + n + 2 \\ &= n^2 + 3n + 2 \\ &= n\,(n + 3) + 2 \end{aligned}$$

Similarly Andre connects his arithmetic knowledge of adding fractions, by converting to equivalent fractions, with the

expressions in equation set three.

Equation Set Three: As $\frac{1}{7} + \frac{1}{8} = \frac{8}{7 \times 8} + \frac{7}{7 \times 8} = \frac{15}{56}$ so;

$$\frac{1}{n} + \frac{1}{n+1} = \frac{n+1}{n(n+1)} + \frac{n}{n(n+1)}$$

$$= \frac{2n+1}{n(n+1)}$$

Andre's achievement exemplifies level five because he recognises that relationships found in equation sets can be expressed using one variable (*n*). This means that he is thinking relationally between the expressions on both sides of the equals sign. Andre connects the generalised processes he has established through arithmetic to the same process operating on the variable *n*. His generalisation of arithmetic extends to operations on fractions and integers as well as whole numbers

Important teaching ideas

The focuses of Algebra, Patterns and Relationships at level 5 are:

i. Generalization of all four arithmetic operations on fractions and integers through the use of variables to show that the generalisations hold for "any number";

ii. Expression of these generalisations through emerging competence with symbolic algebra;

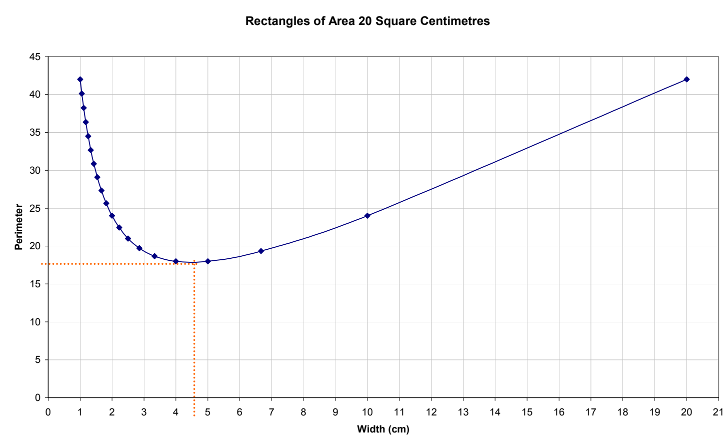
iii. Connection between the representations of linear and simple quadratic relations.

In terms of focus three, it is not intended that the diet of relations and functions that students are exposed to be restricted to linear and simple quadratic forms. Students should experience other types of relations and functions including simple exponential, step and periodic. There is also an expectation that students will explore functions, including domain and range. They should also become familiar with equation and function notation, e.g. $y = 3x + 2$ and $f(x) = 3x + 2$, $x \geq 0$.

A relation is defined as a mapping that connects the elements in two sets. The mapping may be represented as a set of ordered pairs. A function is a special type of a relation, in that, each value of the independent variable is mapped to only one value of the independent variable.

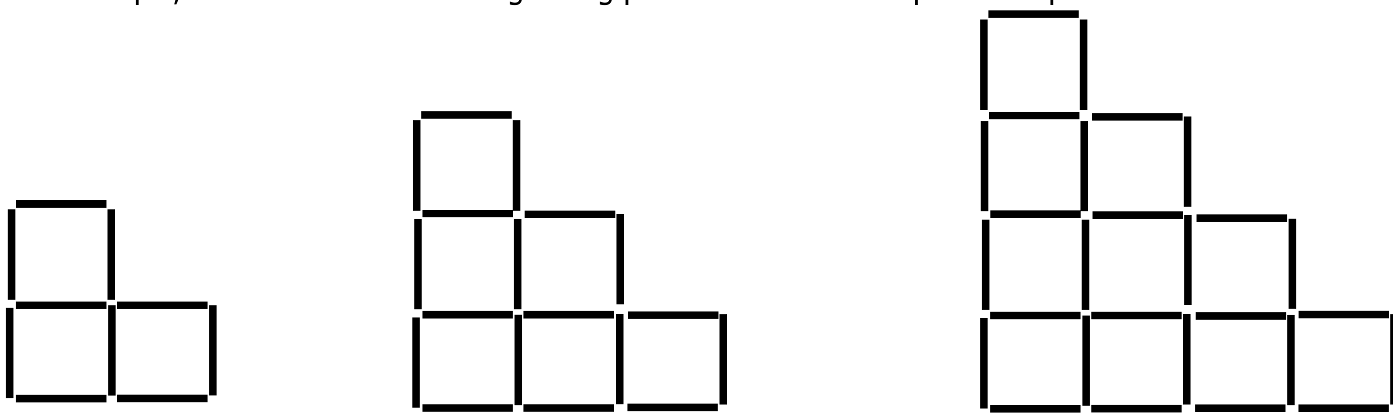
For example, the set of ordered pairs that represent the equation $x^2 + y^2 = 25$ form a circle. This is an example of a relation that is not a function, since there is not a unique y-value for each value of x, e.g. for $x = 0$, $y = {}^-5$ or ${}^+5$. However, the equation $y = 3x + 2$ represents a function since each value of x maps to a unique value of y.

Many of the contexts through which relations and functions are developed involve discrete mappings. For example, through investigating a sequential pattern involving sticks or cubes students may establish a relationship between discrete terms in the sequence, e.g. shape 1, shape 2, etc., and the number of sticks or cubes. It is important for later understanding of functions and graphs that continuous situations be explored. Contexts for doing so usually involve measurement data and a requirement to find values in the range for non-integral values in the domain. For example, students exploring the minimum perimeter of a rectangle of area 20 cm^2 will find a solution to a certain degree of accuracy, e.g. equal sides of 17.89cm (2dp.), and students estimating the length of shadow of a flagpole at 11:42 am. from data collected every hour will need to interpolate (estimate within known values of the data). Using graphs to obtain visual representations of solutions further develops the concept of continuity.



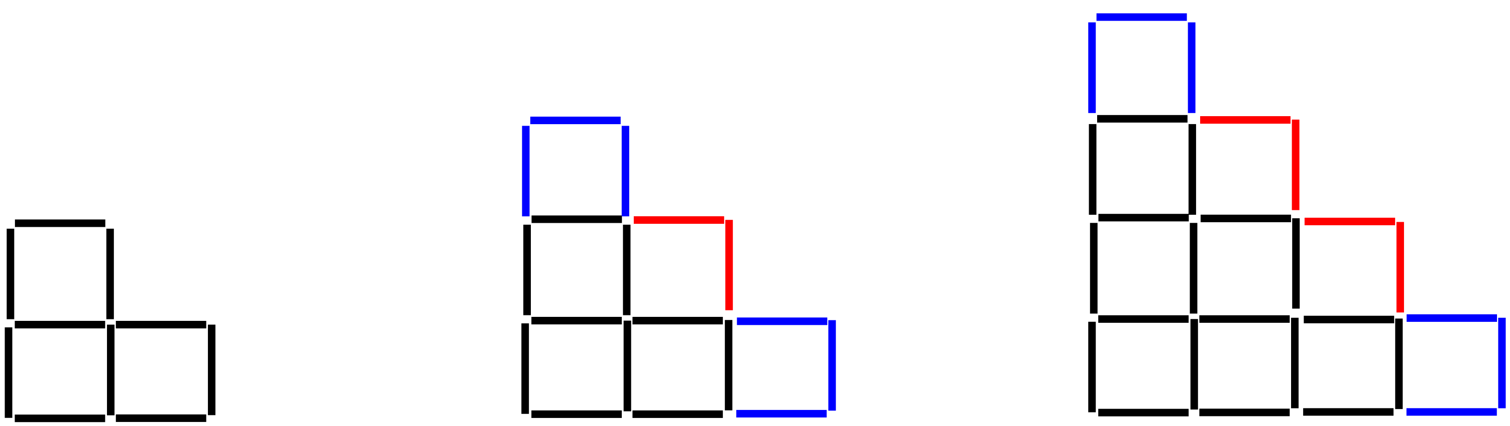
Generalisation involves these progressions (Mason, 2003):

1. **Gazing at the whole**
   For example, a student is shown this growing pattern of stairs. The pattern captures their attention.



2. **Discerning details**
   The student recognizes that the pattern is composed of matchsticks in a square-like array. He/she discerns elements that compose the patterns such as whole squares, L-shapes, and U-shapes. The student may also count numbers of steps and matches.

3. **Recognising relationships**
   The student notices that the height of the staircase is increasing by one step each time and end U-shapes and central L-shapes make up the element that is added each time. This element is increasing by one L-shape (two matches) each time.



4. **Perceiving properties**

The student sees that, given the growth pattern above, a table of values for the relation should show a pattern in the differences between terms. This pattern will allow him/her to predict further members of the pattern without having to build the structure with matches.

Height of staircase	2	3	4	5	*n*
Number of matches	10	18	28	40?	?

+ 8   + 10   + 12

5. **Reasoning on the basis of the properties alone**

The student connects the table for the staircase pattern with that for square numbers knowing that that relation also has a sequence of second order differences that grows by two:

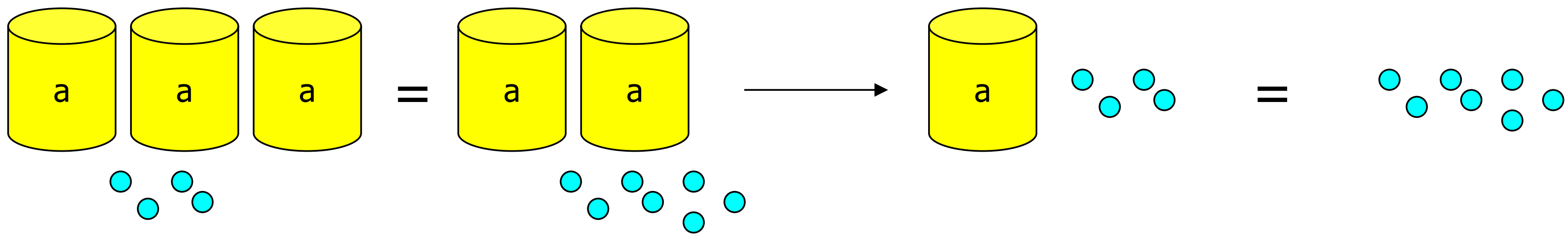
Height of staircase	2	3	4	5	*n*
Number of matches	10	18	28	40	?
	+ 6	+ 9	+ 12	+ 15	
Square numbers	4	9	16	25	n^2

He/she notes that the difference between h^2 and the number of matches is three times the height of the staircase ($3h$). So the student reasons that the formula for the relation is $m = h^2 + 3h$ which can also be expressed as $m = h(h+3)$.

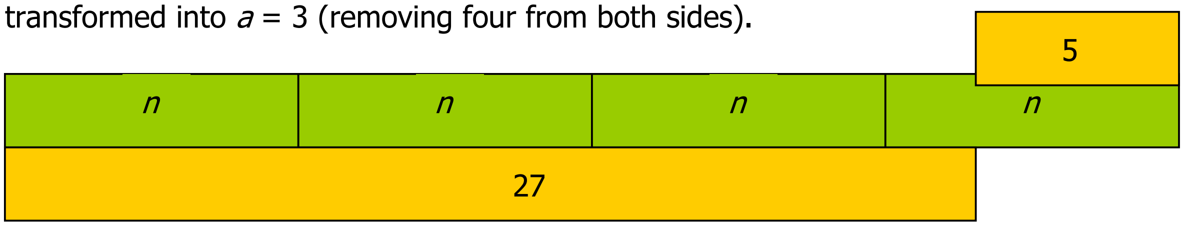
Similar phases in the generalisation process occur in students' experience with operations on whole numbers, fractions and integers. For generalisation to occur, students initially need to connect the symbols with transformations performed on appropriate materials. These materials may be physical objects such as fraction strips or counters, or they may be pictures or diagrams. Generalisation of the number properties requires students initially to anticipate the result of transformations on the materials using the symbolic equations then to operate on the equations without access to materials.

Students at level five should also generalise simple properties of powers though extension of these properties to include fraction and negative integer exponents is not formally required until level six. The use of letters to represent variables in these generalisations, and as specific unknowns where appropriate, needs to be explicitly taught at level five. Algebraic symbolism is a convention. Students frequently invent their own language of symbolism which is easily refined through telling and modeling the correct syntax.

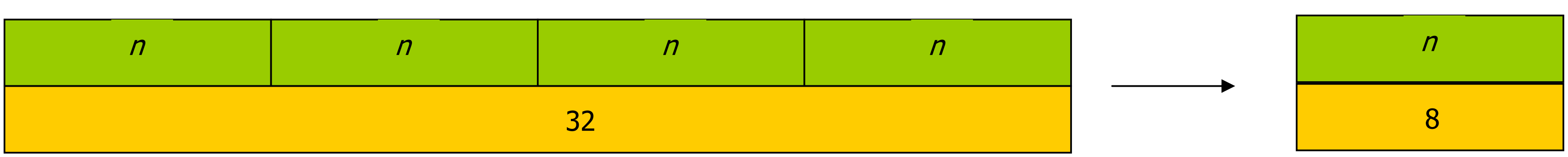
Considerations in teaching the syntax of algebra are students' understanding of the equals sign as a statement of transitive balance, possible confusions between previous number notations and algebraic symbols, the variable uses to which letters are put and the constrained arithmetic strategies that are permitted in algebra. From arithmetic students frequently interpret the equals signs as "the answer follows", and in some instances this is a legitimate use for it. However, in algebra equals signals equality. Certain transformations are able to be performed to the expressions on both sides of an equation while preserving that equality. For example, subtracting the same number or variable from both sides preserves equality as does dividing both sides by the same number.



A cups and counters model of $3a + 4 = 2a + 7$ that can be transformed into $a + 4 = 7$ (removing $2a$ from both sides) then transformed into $a = 3$ (removing four from both sides).



A strips model of $4n - 5 = 27$ that can be transformed into $4n = 32$ (adding five to both sides) then to $n = 8$ (dividing by four).



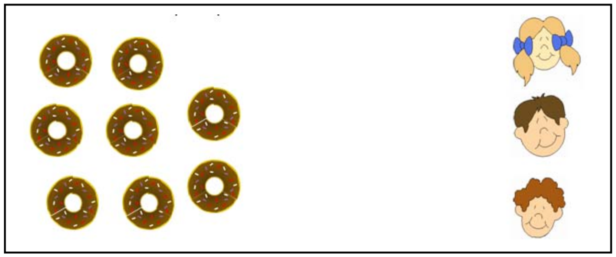
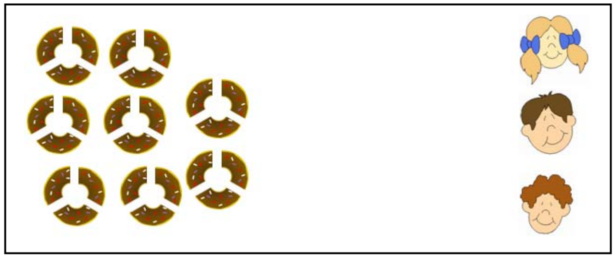
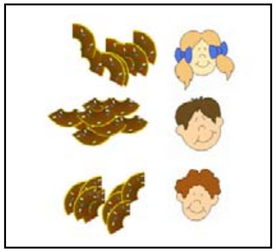
Arithmetic notation can create misconceptions as it is applied to algebra. For example, in arithmetic 64 means six tens and four ones. In algebra $6a$ means six times the variable a. Some students interpret $6a$ as sixty-a. In arithmetic, three divided by twelve is written as 3 ÷ 12 = , and the answer $\frac{3}{12}$ or $\frac{1}{4}$ is a number. In algebra $\frac{c}{d}$ represents a generalised number and the operation $c \div d =$. To make matters more complicated in algebra $\frac{c}{d}$ can be treated as something to be operated on, e.g. $3\frac{c}{d}$ (three times $\frac{c}{d}$), without the closure of completing the operation of division. In arithmetic, 4.7 means four ones and seven tenths. In algebra $3f.4f = 12f^2$. The point means multiplication in algebra which explains the preference in European countries for using a comma rather than a decimal point, e.g. 2,65 for 2.65.

Letters convey multiple meanings from the real world and in their uses within mathematics and the sciences. In literacy letters are the components of words. They have one name but can signal many different sounds. Letters are used in association with numbers to label things, e.g. Room 6b or ◦C, and to locate positions, e.g. C4 locates a square on a map. In mathematics letters are used in several ways. The letters s, t, and r, commonly represent speed, time and radius in formulae. Each of these letters represents a variable and can take up any value. When equations are constrained, e.g. $6m - 7 = 29$, then letters represent specific unknowns. Some letters like i, e, c, and π are used in formulae to represent specific numbers, i.e. $\sqrt{-1}$, base of natural logarithms (2.718…), speed of light, and the ratio of circumference to diameter of a circle (3.14…). It is important that these conventions are explained to students and not assumed as common currency.

Another difference between symbolic algebra and arithmetic is freedom of choice in strategy. In arithmetic problems there usually are several strategies that can be used to solve a problem. For example, to solve 90 ÷ 6 = □, a student might use the distributive property, e.g. 60 ÷ 6 + 30 ÷ 6 = 15, or the associative property, (90 ÷ 3) ÷ 2 = 15, halve both dividend and divisor, 45 ÷ 3 = 15, or resort to an additive method such as 90 – 6 – 6 … However, in simplifying the algebraic expression $\frac{90a}{6b}$ the student must accept the constrained efficient method of looking for a common factor six, and dividing both the numerator and denominator by the common factor, i.e. $\frac{90a}{6b} = \frac{15a}{b} \times \frac{6}{6} = \frac{15a}{b}$. To solve the arithmetic problem, 42 out of 60 as a percentage, the student might choose to reduce the fraction $\frac{42}{60}$ to a simplified form $\frac{7}{10}$, or find a scalar that maps 60 onto 100, or calculate the percentage equivalent to one shot, $100 \div 60 = 1.\dot{6}$. To solve $\frac{42}{k} = 0.7$ in algebra requires the application of a set sequence of efficient procedural steps, i.e. multiply both sides by k then divide both sides by 0.7.

Generalised arithmetic offers excellent opportunities for students to use algebraic thinking. Below are key generalisations that should emerge from arithmetic at level five. The material models are shown to suggest one way in which each concept could be introduced. Using True/False statements as a focus for debate have been shown by research to develop students' ability to define, justify and generalise from specific examples.

1. **Fraction as quotient (answer to division)**
   Example; $8 \div 3 = \frac{8}{3}$

7 donuts shared among 3 students… each donut is cut into thirds… each student gets 7 thirds

True or False?: For eight boys to get as much pizza each as five girls get each, the boys will need to get $\frac{8}{5}$ times the total amount of pizza that the girls get. (Assuming that both the boys and girls share the pizza equally in their groups) (True)

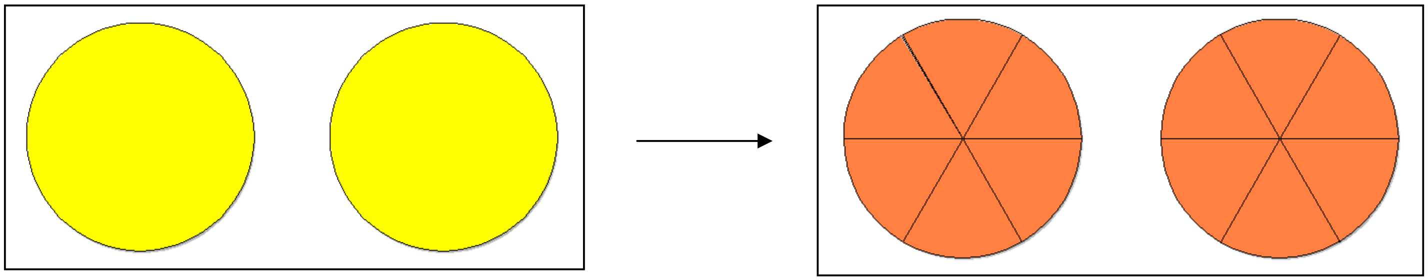
Algebraic substitution examples: What numbers will *w*, *v*, *m*, and *n* be to make the equations true?

$w \div 7 = \frac{6}{7}$ $8 \div v = \frac{8}{6}$ $3 \div 7 = \frac{3}{m}$ $9 \div 4 = \frac{n}{4}$

Algebraic generalisation: $a \div b = \frac{a}{b}$

2. Solving equations involving fractions

Example: You eat twelve pieces and that is two whole pizzas. How much of a pizza is each piece?



As multiplication/division; 12 x Δ = 2, so 6 x Δ = 1, so Δ = 1 ÷ 6, so Δ = $\frac{1}{6}$.As fractions; $\frac{12}{\Delta}$ = 2, so $\frac{6}{\Delta}$ = 1, so Δ = 6, or $\frac{12}{\Delta}$ = 2, so 12 = 2 x Δ, so Δ = 12 ÷ 2, so Δ = 6.

True or False?: You sign up for a cellphone plan for a fixed amount each month. You must stay on the plan for *m* months. The first five months cost \$120 in total. So the total cost for the plan will equal 5 x \$$\frac{120}{m}$. (False. It should be \$120 x $\frac{m}{5}$).

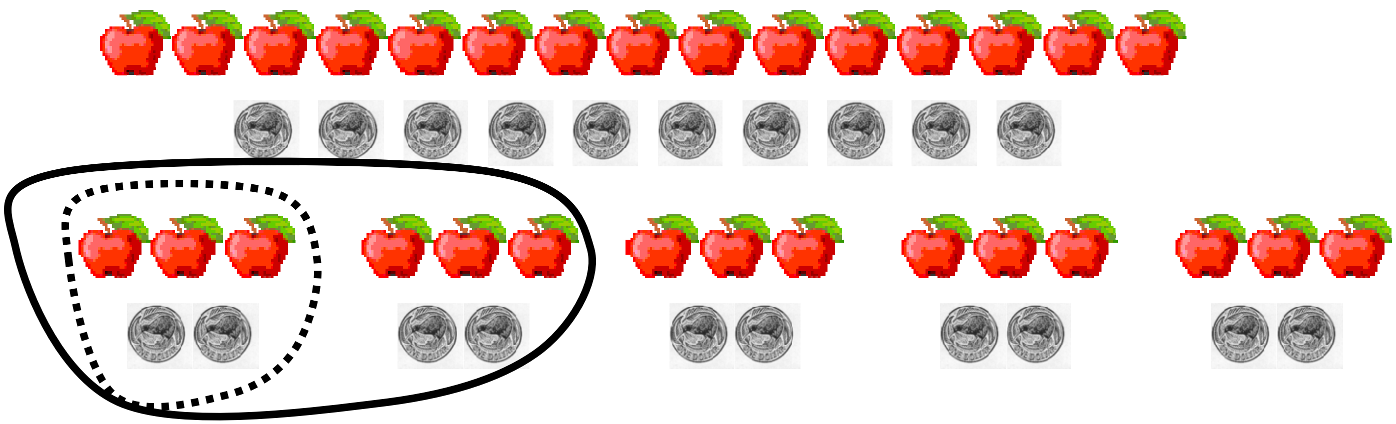
Algebraic substitution examples: What numbers would *p*, *m*, *k*, and *l* be to make the equations true?

$\frac{24}{2} = p$ $\frac{18}{3} = m$ $\frac{16}{k} = 4$ $\frac{l}{6} = 5$

Algebraic generalisation: Given $\frac{a}{b} = c$, then a = *bc* and $b = \frac{a}{c}$

3. **Equal rates or ratios**

Example: 15 apples cost $10. What is the price of 6 apples?



If 15 apples cost $10, then 3 apples cost $2, so 6 apples cost $4 (also 1 apple costs $\$\frac{2}{3}$).

As rates: 15:10 = 3:2 → 3:2 = 6:4

As proportional constants: $\frac{15}{10} = \frac{3}{2} \rightarrow \frac{3}{2} = \frac{6}{4}$ (The constant of proportionality is the multiplier comparing the quantity of one measurement with the quantity of the other measurement, e.g. There are $\frac{3}{2}$ apples for every dollar.)

True or False?: If 25 apples cost $14 then the cost per apple is $\$\frac{14}{25}$. That is $0.56 per apple. (True)

Algebraic substitution examples: What numbers would *k*, *p*, *q*, *b*, *z*, *y*, *r and s* represent to make the equations true?

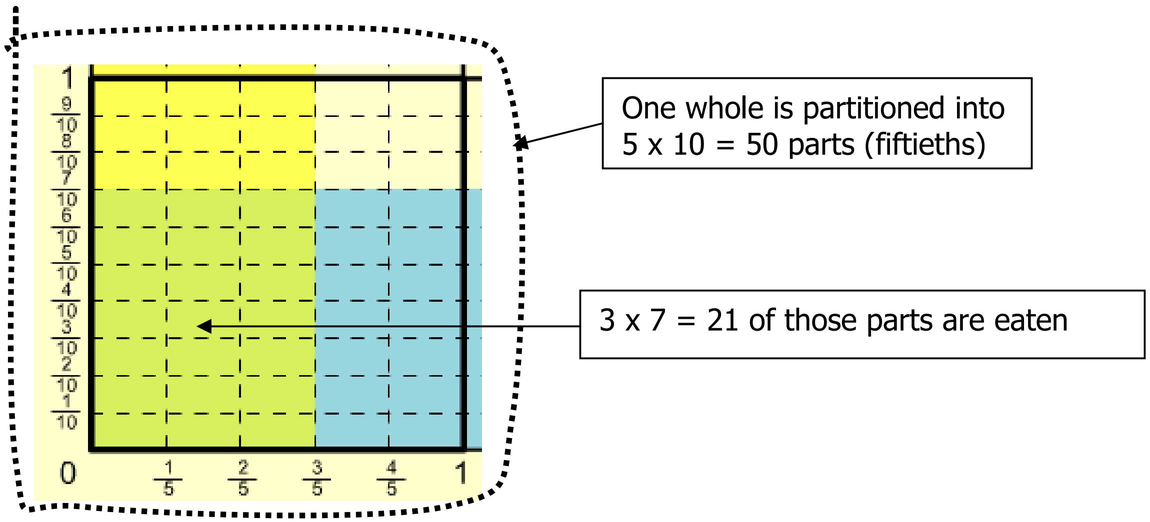
14: k = 28: 16	p :15 = 8:20	5:8 = q:24	18:12 = 3:b
$\frac{5}{7} = \frac{35}{z}$	$\frac{11}{3} = \frac{y}{9}$	$\frac{16}{r} = \frac{12}{21}$	$\frac{s}{10} = \frac{35}{45}$

Algebraic generalisation: Given two equal rate/ratio pairs *a:b* and *c:d*, then $\frac{a}{b} = \frac{c}{d}$ (and $ad = bc$).

4. **Multiplication of fractions (and decimals)**

Example: You eat three-fifths of seven-tenths of a chocolate block. How much of one block do you eat?

$\frac{3}{5} \times \frac{7}{10} = \frac{21}{50}$



True or False: The answers to $\frac{3}{5} \times \frac{4}{9}$ and $\frac{4}{5} \times \frac{3}{9}$ are the same but the answer to $\frac{4}{3} \times \frac{9}{5}$ is different. (true)

Algebraic substitution examples: What numbers do *b*, *r*, *x*, *y*, *h*, *I and j* represent to make the equations true?

$\frac{7}{13} \times \frac{2}{3} = \frac{b}{39}$ $\frac{5}{c} \times \frac{8}{13} = \frac{40}{91}$ $\frac{7}{14} \times \frac{d}{10} = \frac{35}{140}$ $\frac{7}{5} \times \frac{4}{12} = \frac{7}{e}$

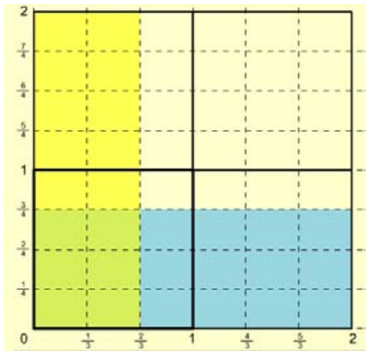
$0.2 \times 0.8 = h$ $2.3 \times i = 0.23$ $j \times 0.045 = 9.0$

Algebraic Generalisation: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

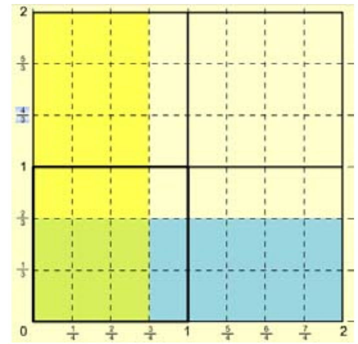
5. Commutative property for multiplication of fractions (and decimals)

Example: Which is greater two-thirds of three-quarters of a pie or three-quarters of two-thirds?

$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$



$\frac{3}{4} \times \frac{2}{3} = \frac{6}{12}$



True or False?: When you multiply two fractions the order makes no difference to the answer, just like with whole numbers, e.g. 4 x 6 = 6 x 4 so $\frac{2}{3} \times \frac{5}{7} = \frac{5}{7} \times \frac{2}{3}$ (True)

Algebraic substitution examples: What numbers would *b*, *r*, *x*, *and y* represent to make the equations true?

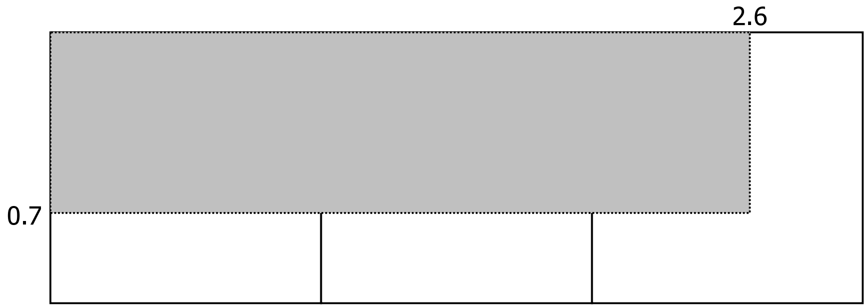
$\frac{11}{113} \times \frac{3}{17} = \frac{b}{17} \times \frac{11}{113}$ $\frac{4}{r} \times \frac{8}{19} = \frac{8}{19} \times \frac{4}{9}$ $\frac{7}{15} \times \frac{11}{14} = \frac{7}{14} \times \frac{x}{15}$ $\frac{12}{16} \times \frac{y}{23} = \frac{13}{23} \times \frac{24}{16}$

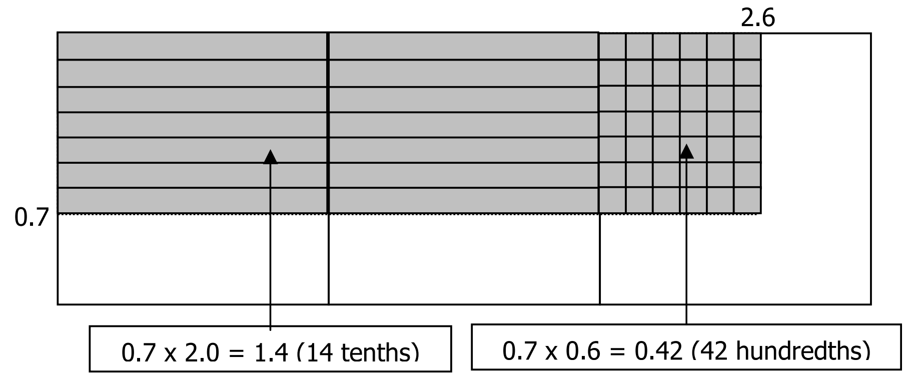
0.25 x k = 9.45 x 0.5 1.67 x 0.08 = m x 1.67

Algebraic Generalisation: $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$ or $\frac{ac}{bd} = \frac{ca}{db}$

6. Distributive property for multiplication of fractions (and decimals)

Example: You want to plant grass seed on an area of dirt that is 0.7m x 2.6m. What is the area of the dirt in m^2?





So 0.7 x 2.6 = (0.7 x 2.0) + (0.7 x 0.6)
= 1.4 + 0.42
= 1.82

True or False?: 0.9 x 2.3 has the same answer as 1.0 x 2.2. (False)

Algebraic substitution examples: What numbers would *b*, *c*, *d*, *e*, *f*, *g*, *h*, and *i* represent to make the equations true?

$b \times 0.9 = 3 \times 0.9 + 0.4 \times 0.9$ $\quad 2.3 \times 0.4 = c \times 0.4 + 0.3 \times 0.4$ $\quad 6.5 \times 0.8 = 6 \times 0.8 + d \times 0.8$

$3.2 \times 5.6 = 3 \times 5.0 + 0.2 \times 5.0 + e \times 5.0 + 0.2 \times 0.6$ $\quad 0.79 \times 2.3 = 0.7 \times 2.0 + f \times 0.3 + 0.09 \times 2.0 + 0.09 \times 0.3$

$\frac{5}{10} \times \frac{23}{10} = \frac{5}{10} \times \frac{2}{1} + \frac{5}{10} \times \frac{g}{10}$ $\qquad \frac{7}{100} \times \frac{43}{10} = \frac{7}{100} \times \frac{h}{1} + \frac{7}{100} \times \frac{3}{10}$

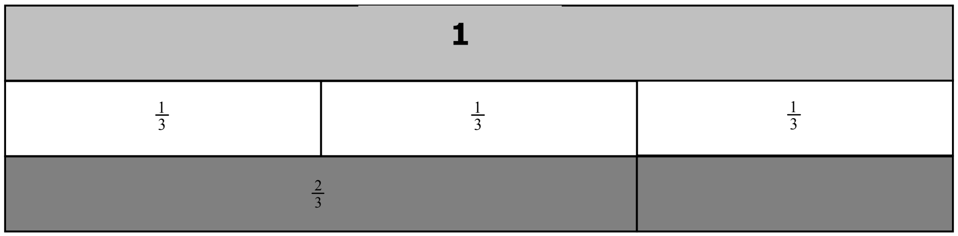
$\frac{84}{10} \times \frac{36}{100} = \frac{8}{1} \times \frac{3}{i} + \frac{8}{1} \times \frac{6}{100} + \frac{4}{10} \times \frac{3}{10} + \frac{4}{10} \times \frac{6}{100}$

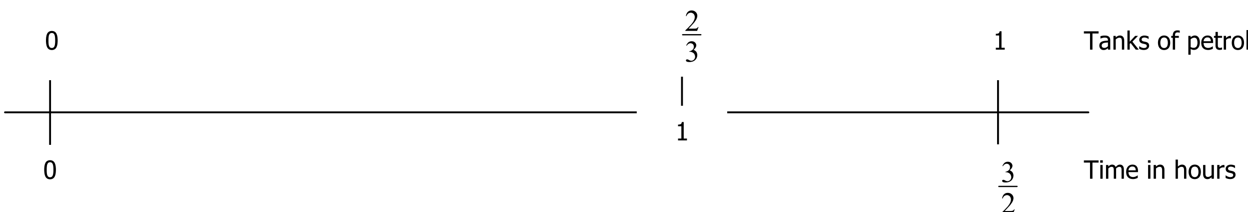
Algebraic generalisation: $\frac{a}{b} \times \frac{c}{d} = \frac{a}{b} \times \frac{e}{d} + \frac{a}{b} \times \frac{f}{d}$, If $c = e + f$.

7. Division by fractions

Example:
- a. Your lawnmower uses two-thirds of a tank of petrol each hour. How long will it run on a full tank?
- b. You have enough petrol to fill the tank four times. How much mowing time will that give you?
- c. How long will the mower run on one-half of a tank?

a.

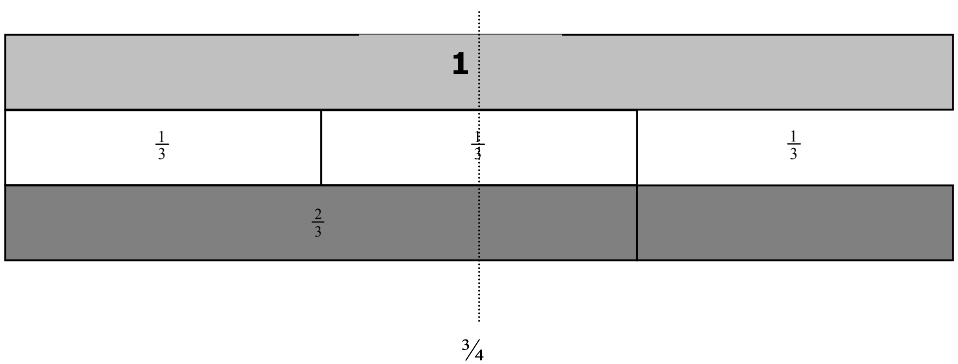


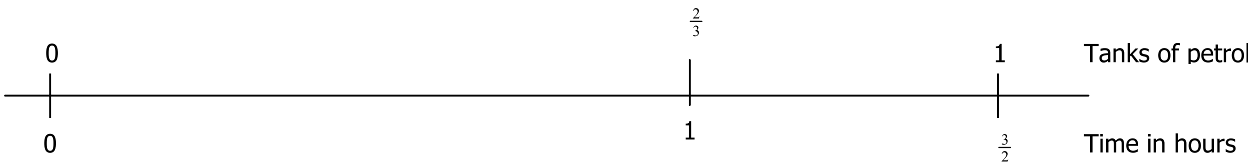


$1 \div \frac{2}{3} = \frac{3}{2}$ and $1 \times \frac{3}{2} = \frac{3}{2}$,

b. So $4 \div \frac{2}{3} = \frac{12}{2}$ and $4 \times \frac{3}{2} = \frac{12}{2}$ (Scaled up by four)

c. So $\frac{1}{2} \div \frac{2}{3} = \frac{3}{4}$ and $\frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$. (Scaled down from a. by one half)





True or False?: $\frac{5}{8} \div \frac{3}{4} = \square$ has the same answer as $\frac{3}{4} \div \frac{5}{8} = \square$. (False)

Algebraic substitution examples: What numbers would *n*, *m*, *r* and *p* represent to make the equations true?

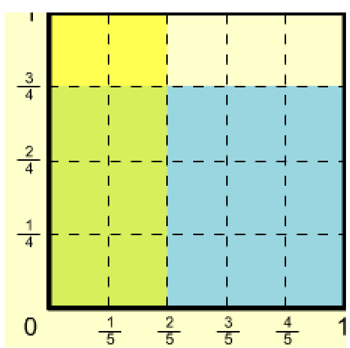
$1 \div \frac{4}{5} = \frac{5}{n}$ $\qquad$ $\frac{2}{3} \div \frac{7}{11} = \frac{2}{3} \times \frac{m}{7}$ $\qquad$ $\frac{2}{r} \div \frac{4}{9} = \frac{18}{36}$ $\qquad$ $\frac{5}{7} \div p = \frac{5}{7} \times \frac{2}{3}$

Algebraic generalisation: $1 \div \frac{a}{b} = \frac{b}{a}$ and $1 \times \frac{b}{a} = \frac{b}{a}$, so $n \div \frac{a}{b} = \frac{nb}{a}$ and $n \times \frac{b}{a} = \frac{nb}{a}$, $\frac{c}{d} \div \frac{a}{b} = \frac{cb}{da}$ and $\frac{c}{d} \times \frac{b}{a} = \frac{cb}{da}$

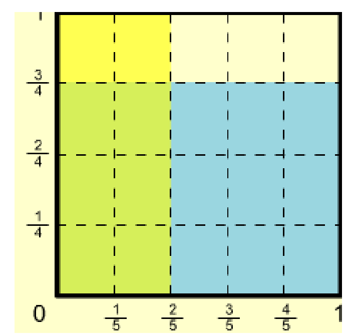
8. Multiplication and division as inverses (with fractions)

Example: a. In the fridge two-fifths of a chocolate bar is left. You eat three-quarters of it. How much of a whole bar do you eat?
b. Last night you ate six-twentieths of a whole chocolate bar. You started with a piece that was two-fifths of a bar. What fraction of that two-fifths piece did you eat?

a. $\frac{3}{4} \times \frac{2}{5} = \frac{6}{20}$



b. $\frac{6}{20} \div \frac{2}{5} = \frac{3}{4}$



True or False?: If $\frac{4}{9}$ x $\frac{3}{5}$ = $\frac{12}{45}$ then $\frac{4}{15}$ ÷ $\frac{4}{9}$ = $\frac{3}{5}$. (true)

Algebraic substitution examples: What numbers would *v*, *w*, *x*, *y*, and *z* represent to make the equations true?

$\frac{3}{2}$ x $\frac{3}{7}$ = $\frac{9}{14}$ so $\frac{9}{14}$ ÷ $\frac{3}{7}$ = $\frac{63}{42}$ = $\frac{v}{2}$ $\frac{3}{4}$ ÷ $\frac{7}{11}$ = $\frac{33}{28}$ so $\frac{7}{11}$ x $\frac{33}{28}$ = $\frac{3}{w}$ $\frac{7}{11}$ x $\frac{3}{5}$ = $\frac{21}{55}$ so $\frac{21}{55}$ ÷ $\frac{x}{11}$ = $\frac{3}{5}$

$\frac{1}{4}$ ÷ $\frac{3}{8}$ = $\frac{8}{12}$ = $\frac{2}{3}$ so $\frac{3}{8}$ x $\frac{y}{9}$ = $\frac{1}{4}$ $\frac{3}{5}$ x $\frac{1}{7}$ = $\frac{3}{35}$ so $\frac{3}{35}$ ÷ $\frac{6}{z}$ = $\frac{1}{7}$

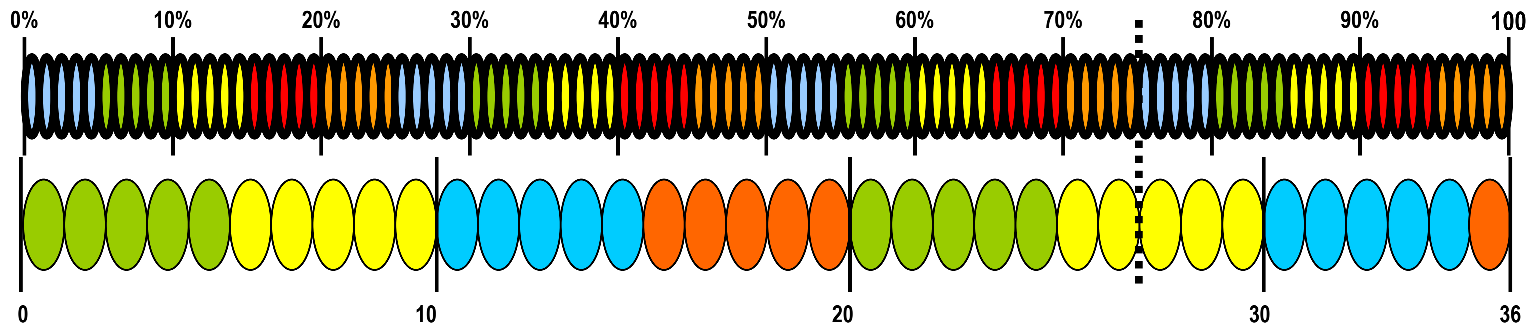
Algebraic generalisations: $\frac{a}{b}$ x $\frac{c}{d}$ = $\frac{ac}{bd}$ so $\frac{ac}{bd}$ ÷ $\frac{c}{d}$ = $\frac{a}{b}$ and $\frac{ac}{bd}$ ÷ $\frac{a}{b}$ = $\frac{c}{d}$

$\frac{ac}{bd}$ ÷ $\frac{a}{b}$ = $\frac{c}{d}$ so $\frac{ac}{bd}$ ÷ $\frac{c}{d}$ = $\frac{a}{b}$ and $\frac{a}{b}$ x $\frac{c}{d}$ = $\frac{ac}{bd}$

9. Percentages as equivalent fractions and fraction operators

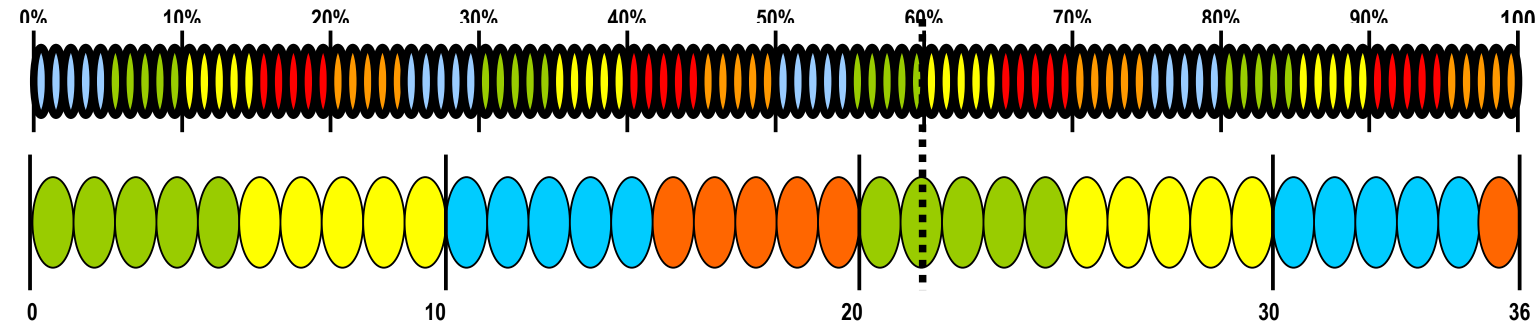
Example:

a. In tennis you practise 36 first serves. Twenty-seven of your serves are called good. What is your serving percentage?



$\frac{27}{36}$ = $\frac{3}{4}$ = $\frac{75}{100}$ = 75%

b. You pay only 60% of the price. It usually costs $36.00. What do you pay?



60% of 36 = □ is $\frac{60}{100}$ x 36 = $\frac{60 \times 36}{100}$ = 21.6 ($21.60) (Note that $\frac{a}{b}$ = a ÷ b is assumed)

True or False?: 42% of 78 = □ has the same answer as 78% of 42 = □. (true)

Algebraic substitution examples: What numbers would *v*, *w*, *x*, *y*, and *z* represent to make the equations true?

$\frac{18}{25}$ = *q*% $\frac{a}{40}$ = *55*% $\frac{45}{27} = \frac{c}{100}$ $\frac{14}{j} = \frac{20}{100}$

80% of 35 = $\frac{h}{10}$ x 35 *b*% of 79 = $\frac{63}{100}$ x 79 $\frac{35}{100}$ x 20 = $\frac{f}{100}$ $\frac{z}{100}$ x 24 = $\frac{72}{100}$

10. Addition and subtraction with fractions

Example: a. In the shed there are two pails of the same paint. One pail is three-quarters full and the other is two-thirds full. How much of one pail is that altogether?

$$\frac{3}{4} + \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{17}{12}$$

$\frac{1}{4}$			$\frac{1}{4}$			$\frac{1}{4}$			$\frac{1}{3}$				$\frac{1}{3}$			
$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
1												$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

b. It will take one and three-fifths of a pail to paint the roof. You have two-thirds of a pail. How much more paint do you need?

$\frac{8}{5} - \frac{2}{3} = \frac{24}{15} - \frac{10}{15} = \frac{14}{15}$ (Note subtraction is seen as difference)

1															$\frac{1}{5}$			$\frac{1}{5}$			$\frac{1}{5}$		
$\frac{1}{3}$					$\frac{1}{3}$					?													
$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$

True or False?: If $\frac{2}{5} + \frac{3}{7} = \frac{29}{35}$ then $\frac{29}{35} - \frac{3}{7} = \frac{2}{5}$. (True)

Algebraic substitution examples: What numbers would *d*, *k*, *m*, and *p* represent to make the equations true?

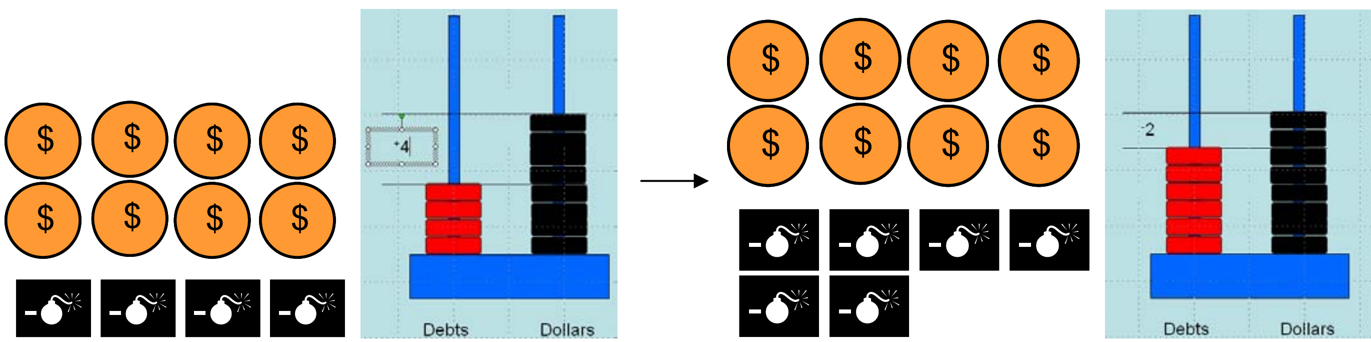
$\frac{5}{8} + \frac{3}{7} = \frac{d}{56}$ $\qquad$ $\frac{9}{11} - \frac{3}{4} = \frac{k}{44}$ $\qquad$ $\frac{3}{13} + \frac{m}{2} = \frac{45}{26}$ $\qquad$ $\frac{13}{p} - \frac{4}{5} = \frac{49}{20}$

Algebraic generalisations: $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad+cb}{bd}$, and $\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{cb}{bd} = \frac{ad-cb}{bd}$

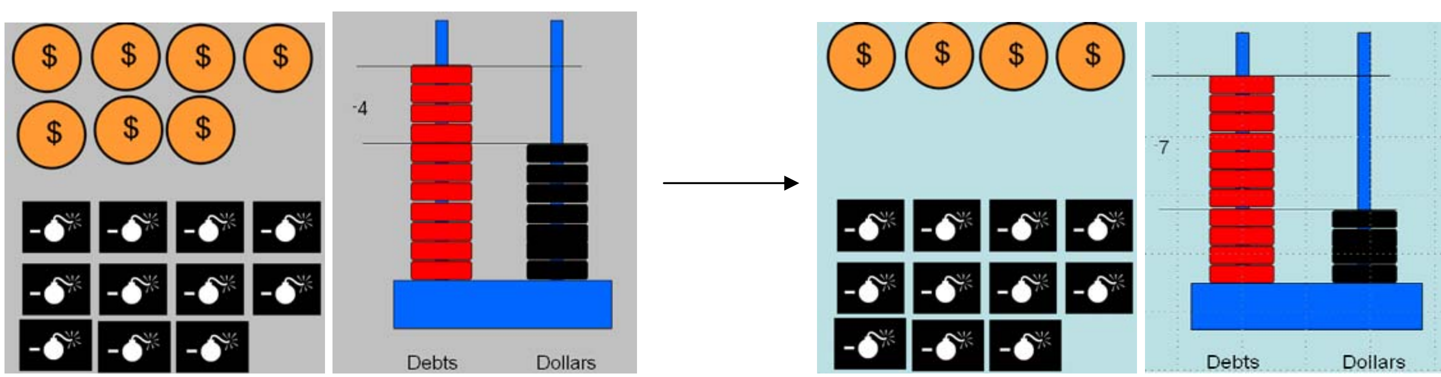
Note: The commutative, associative and inverse properties of addition and subtraction as they relate to fractions should also be explored.

11. Addition and subtraction of integers

Example: a. You have \$8 in cash and \$4 in debts. How much are you worth? How much will you be worth if you add another \$2 in debt? (${}^{+}4 + {}^{-}2 = {}^{+}2$)



b. You have \$7 in cash but owe \$11 in debts. How much are you worth? How much will you be worth if you spend \$3? (${}^{-}4 - {}^{+}3 = {}^{-}7$)



True or False?: If you subtract any number from 12, the answer is always less than 12. (False)

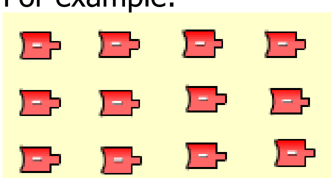
Algebraic substitution examples: These examples are aimed at helping students to recognise the equivalent effects of operations, e.g. ${}^{+}4 + {}^{-}2 = {}^{+}2$ and ${}^{+}4 - {}^{+}2 = {}^{+}2$
What numbers would c, y, r, l and b represent to make the equations true?
${}^{+}4 + {}^{-}2 = {}^{+}4 - c$ $\quad$ $v + {}^{+}6 = {}^{+}6 - {}^{+}3$ $\quad$ ${}^{-}5 - r = {}^{-}5 + {}^{+}7$ $\quad$ ${}^{+}8 - {}^{-}3 = {}^{-}3 + l$ $\quad$ ${}^{-}({}^{+}6 + {}^{+}7) = b + {}^{-}6$

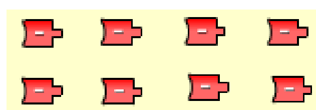
Algebraic generalisations: For any integers a, b, c then:
$a + b = b + a$ (Commutative property); $a + (b + c) = (a + b) + c$ (Associative property);
$a + b = c$, then $c - a = b$ and $c - b = a$ (Inverse operations)

Posing open questions that involve renaming expressions with integers in multiple ways helps students recognise equivalence.
For example:
What expressions are the same as $a - b$, if a and b are integers? [e.g. $a + {}^{-}b$, ${}^{-}b + a$, $-(b - a)$]

12. Multiplication and division of integers

The "rules" for addition, subtraction, multiplication and division of integers arise from extension of patterns that preserve the number properties (commutative, distributive, associative, inverse). Since it is difficult to find helpful real world models for multiplication and division of integers it is wise pedagogically to use pattern in discovering these "rules". Once the generalisations are established they can be applied to model contexts such as transmission factors in gear/pulley trains and multiple enlargements.
For example:





If $3 \times {}^{-}4 = {}^{-}12$ $\quad$ $2 \times {}^{-}4 = {}^{-}8$ $\quad$ $1 \times {}^{-}4 = {}^{-}4$
What will be the answers to; $0 \times {}^{-}4 = \square$, ${}^{-}1 \times {}^{-}4 = \square$, ${}^{-}2 \times {}^{-}4 = \square$, …?
If $3 \times {}^{-}4 = {}^{-}12$ then what are the answers to ${}^{-}12 \div 3 = \square$ and ${}^{-}12 \div {}^{-}4 = \square$?

Algebraic substitution examples: These examples are aimed at helping students to recognise the equivalent effects of operations, e.g. $^-4$ x $^-2$ = $^+8$ and $^+4$ x $^+2$ = $^+8$, and the commutative, associative, distributive properties, and multiplication and division as inverse operations.

What numbers would the letters represent to make the equations true?

$^+6 \times {}^-3 = {}^+3 \times s$ $\quad$ $^-6 \times a = {}^+7 \times {}^+6$ $\quad$ $^-6 \times ({}^-3 \times {}^+8) = {}^-48 \times w$ $\quad$ $^-88 \times {}^-66 = {}^+44 \times (p \times {}^+33)$

$h \times {}^+7 = {}^+7 \times {}^+2 + {}^+7 \times {}^-40$ $\quad$ $^-14 \times {}^+27 = {}^-378$, so $^-378 \div {}^+27 = d$

$^-12 \div {}^+3 = k \div {}^-3$ $\quad$ $^+201 \div z = {}^-402 \div {}^-6$

$^-24 \times {}^+6 = {}^-144$ so $^-144 \div {}^-24 = v$ $\quad$ $^+1701 \div {}^-63 = {}^-27$ so $j \times {}^+27 = 1701$

True or false?: All of these multiplication problems have the same answer; $^+4 \times {}^-5 = \square$, $^+5 \times {}^-4 = \square$, $^-4 \times {}^+5 = \square$, $^-5 \times {}^+4 = \square$. (true)

Algebraic generalisations: The "truncation" of symbols for multiplication should be introduced as a convention to reduce recording, e.g. 6 x *a* as 6*a*, *b* x *c* as *bc*. Address the potential confusion with place value, e.g. 6*a* might be confused with "sixty-*a*".

With division the truncation $a \div b = \frac{a}{b}$ requires students accept the fraction as quotient generalisation (as above).

For any integers *a*, *b*, *c* then:

$ab = ba$ (Commutative property); $a(bc) = (ab)c$ (Associative property); $a(b + c) = ab + ac$ (Distributive property);

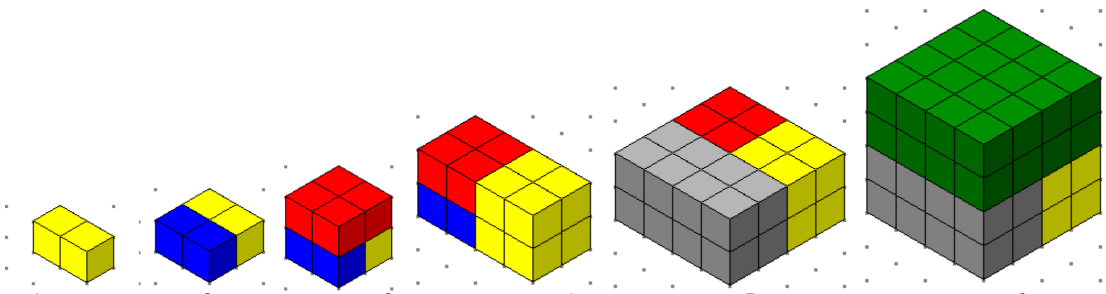
$ab = c$ then $\frac{c}{a} = b$ and $\frac{c}{b} = a$ (Inverse operations)

13. Multiplication and division involving powers (using exponents)

Example: Use this table to help you calculate:

a. $32 \times 64 = \square$ b. $16 \times 128 = \square$ c. $1024 \div 16 = \square$ d. $512 \div 512 = \square$

A table of powers of two is built up using connecting cube models held together by rubber bands. This works adequately until 2^9 = 512 and shows that the model forms a cube for every exponent that is a multiple of three.



2^1 2^2 2^3 2^4 2^5 2^6

A table for powers of two is created and extended, possibly using a spreadsheet (=n^2 as, ^ means "to the power of" in spreadsheet syntax).

	A	B	C	D	E	F	G	H	I	J	K	L
1	Number (n)	1	2	3	4	5	6	7	8	9	10	11
2	2^n	2	4	8	16	32	64	128	256	512	1024	2048

a. 32 x 64 = 2048
$2^5 \times 2^6 = 2^{11}$

b. 16 x 128 = 2048
$2^4 \times 2^7 = 2^{11}$

c. 1024 ÷ 16 = 64
$2^{10} \div 2^4 = 2^6$

d. 512 ÷ 512 = 1
$2^9 \div 2^9 = 2^0$

In order to generalise multiplication and division by powers students need experience with different numbers (logarithmic bases), e.g. powers of three and five. Recognition that the properties hold for fractions (bases), e.g. $\frac{1}{3}^4$ prepares students for level six in which powers are extended to include exponents that are integers and fractions, e.g. $3^{1/4}$ = □, 7^{-2} = □.

True of False?: 2^4 = □ has the same answer as 4^2 = □ but 3^4 = □ does not have the same answer as 4^3 = □. (True)

Algebraic substitution examples:
What numbers do the letters represent to make the equations true?

$3^4 \times 3^a = 3^8$ $7^b \times 7^2 = 7^{11}$ $6^4 \times 6^4 = 6^c$ $6^7 \div 6^4 = 6^d$ $8^f \div 8^2 = 8^6$ $0.5^{12} \div 0.5^h = 0.5^3$

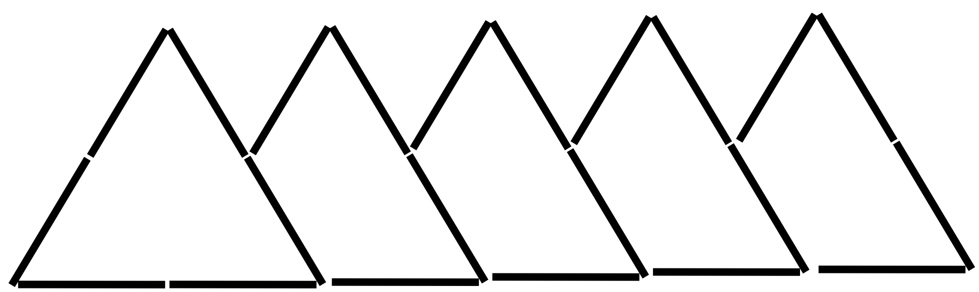
Algebraic generalisations:
What numbers do the letters represent to make the equations true?

$n^a \times n^b = n^c$ *so* $n^c \div n^a = n^b$ and $n^c \div n^b = n^a$

Patterns Based Approaches

Apart from generalised arithmetic approaches students should have experiences in finding relationships through geometric patterns and practical experiments (i.e. measurement activities). Geometric patterns have the advantages of connecting geometry with number and algebra, of allowing students to use spatial reasoning, and of developing relationships in constrained, discrete situations. However, geometric patterns can increase the cognitive load of a problem and students need to be taught strategies to ease this load, such as recoding data in tables and beginning relationship seeking through the simplest terms. Practical situations involve measurement and promote the finding of relationships in continuous situations.
It is expected that from level four students will bring competence in connecting representations of linear relations and functions, i.e. table of value, graph, equation.
For example, consider the geometric pattern:

Five "stacked" triangles take 22 sticks to build.



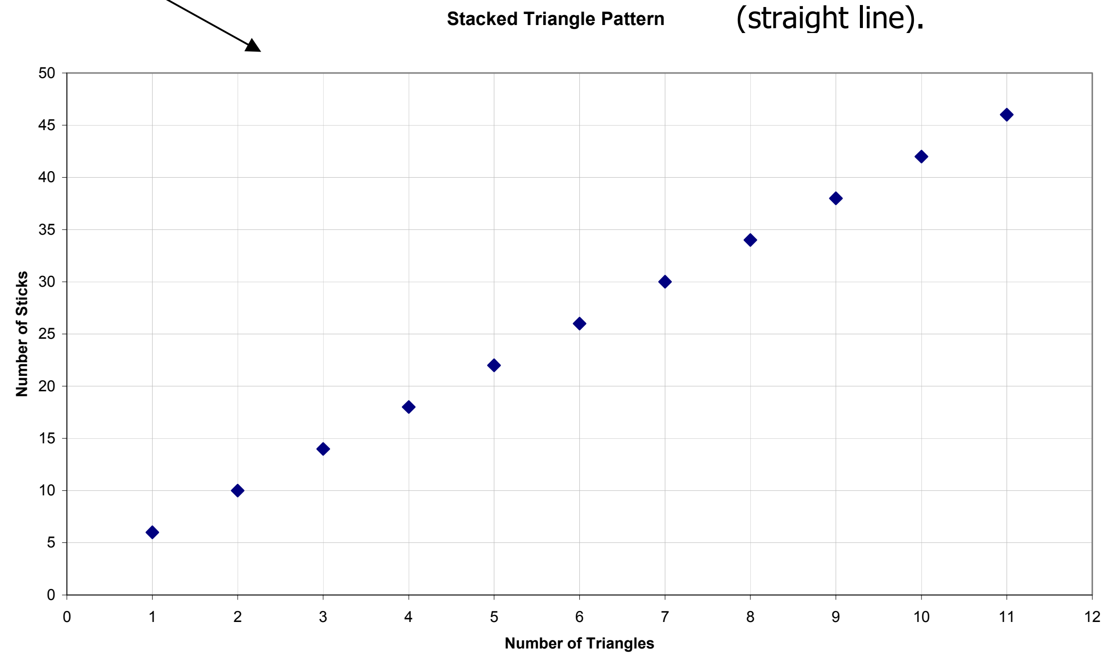
Representations of the relationship between the number of triangles (t) and the number of sticks (s) are:

Triangles	1	2	3	4	5	6	7	8	9	10	11
Sticks	6	10	14	18	22	26	30	34	38	42	46

Ordered pairs are transpositions of the table values into form (t, s). Slope of line connects to constant difference, hence a linear relation (straight line).

Constant difference of four connects to coefficient of four in equation. Constant of two reflects extra sticks required for term one.

$$s = 4t + 2$$



Coefficient of four in equation connects to slope of line through the ordered pairs. Constant of two connects to intercept of that line with vertical axis.

Linear relations in continuous situations can also be developed through practical activities such as finding the relationship between the diameter and circumference of circles by measuring cylindrical objects (π).

Useful resources

Figure It Out (Learning Media)
Algebra Level 3-4, pages 1-24
Algebra year 7/8 Books 2 and 3
Number Sense and Algebraic Thinking, Levels 3-4
Proportional Reasoning Book 1, Levels 3-4+
Proportional Reasoning Book 2, Levels 3-4+
For planning sheets with full references and learning outcomes for the Figure It Out books go to nzmaths:
http://www.nzmaths.co.nz/node/1917

[Comprehensive Teacher notes are provided for each student book. These notes have been distributed to schools and can also be accessed through http://www.tki.org.nz/r/maths/curriculum/figure/index_e.php

Numeracy Project Book 8: Teaching Number Sense and Algebraic Thinking, pages 12, 13, 15-18,21-28,37-47.
Numeracy Project Book 9: Teaching Number through Measurement, Geometry, Algebra, and Statistics, pages 30-40.

nzmaths.co.nz units (This website is sponsored by the Ministry of Education)
http://www.nzmaths.co.nz/node/396 (Algebra: Holistic Algebra)
http://www.nzmaths.co.nz/node/399 (Algebra: Arithmagons)
http://www.nzmaths.co.nz/node/401 (Algebra: Linear Graphs and Patterns)
http://www.nzmaths.co.nz/node/402 (Algebra: Beanies)

Digital Learning Objects (These are accessed through the Ministry of Education Digi-Store and are the result of a collaborative project run by The Learning Federation, Australia)
http://www.nzmaths.co.nz/learningobjects/313/5

Other Website links:
http://illuminations.nctm.org/Activities.aspx?grade=3&grade=4&srchstr=algebra
http://illuminations.nctm.org/Activities.aspx?grade=3&grade=4&srchstr=number
http://nlvm.usu.edu/en/nav/category_g_3_t_2.html

http://nlvm.usu.edu/en/nav/category_g_3_t_1.html
http://nrich.maths.org/public/freesearch.php?search=algebra+stage+4