Mathematics in the New Zealand Curriculum
Second Tier

Strand: Geometry and Measurement　　Thread: Measurement　　Level: Five

Achievement Objectives:

- Use appropriate scales, devices, and metric units for length, area, volume and capacity, weight (mass), temperature, angle, and time.
- Convert between metric units, using whole numbers and commonly used decimals.
- Use side or edge lengths to find the perimeters and areas of rectangles, parallelograms, and triangles and the volumes of cuboids.
- Interpret and use scales, timetables and charts.

Important Teaching Ideas

Developing as a Measurer

The key idea of measurement at **Level One** is **comparison** within a chosen measurement attribute such as length, weight etc. Comparing, and therefore ordering, is the first step in a person becoming a mature measurer. Hence the language of measurement at this stage is ‘longer’, ‘shorter’, ‘smaller’, ‘bigger’, ‘heavier’, ‘lighter’, ‘hotter’, ‘colder’, etc.

Perception of the attributes of measurement (length, weight, time etc.) is fundamental and in fact precedes, or is developed in conjunction with, the comparative concepts. For example the **concept** of distance must be developed before any attempt to measure distance is made. Appropriate use of language such as ‘shorter’ and ‘longer’, ‘further and ‘closer’ will develop and strengthen the concept of distance. The definitions of the measurement attributes of length, area, volume, capacity, weight, angle, temperature and time are given in the glossary. The metric system (SI) is also well covered in the glossary.

An important distinction is between **‘direct comparison’** and **‘indirect comparison’**. As an example, we could compare the lengths of two objects by placing them next to each other and thus observing which is the longer. That would be an example of direct comparison. Alternatively, we could take a piece of string the length of one object and place it against the other object and so compare their lengths that way. That would be an example of indirect comparison. It is important that children experience both forms of comparison at Level One. Indirect comparison is an important step towards the use of non-standard and standard measurement units and the use of measurement tools like rulers and jugs.

Students at level one should generalise the principles of counting to measurement. Counting describes the total number of objects in a set so measuring describes the total number of measures in a space. For example, a student measuring the width of a room using footsteps needs to understand that the count, 1,2,3,.., is describing the number of steps to that point not naming each step.

The key idea of measurement at Level Two is the use of units and 'devices' to measure length, area, volume etc. These units may be standard or non-standard. A non-standard unit is a unit of measurement that has been chosen by a group of people because it is convenient to use. Non-standard usually means that the unit is not universally accepted. Historically non-standard units have usually been decreed by or taken from body parts of an important person like a tohunga. A standard unit is one within the SI (metric) system, e.g. litre, metre, that are universally accepted if not always used, e.g. USA still uses pounds and miles.

So for length, examples of non-standard measures would be paces, hand spans or pencil lengths; for area, exercise books or sheets of newspaper. Non-standard units should be selected from the student's environment and experiences. The use of non-standard units and whole numbers of standard units introduces the student to the concept of measurement units without requiring the rigour of understanding the metric system, particularly the multiplication and division required, e.g. 1000 metres is 1 kilometre. Further it gives the student the opportunity to develop skills of estimation which are required for operating effectively in the adult world.

In measuring the length of, for example, a desk, students can choose an appropriate unit and see how many of those units can be fitted into the length of the desk. Length is a continuous measure but in order to measure the length of a desk we need to partition the length into a sequence of the chosen units, counting how many of those smaller subdivisions will fit into the length of the desk. We may need to partition the unit itself and consider halves and quarters to finish the task. What we call the length of the desk is the combination of those units and part units. Measurement is a critical context for the connection of whole (units) to parts (fractions). Once again, it is the completing of such tasks and the appropriate discussion that is associated that will lead the students to such understandings.

Students at Level Two should be able to apply the addition and subtraction, simple multiplication and division understandings they have from number to measurement problems involving whole numbers of units, e.g. 6 metres. Most children will have encountered some basic metric units, such as metres, kilometres, litres and kilograms, without realising how they relate to one another. For example, they should be able to recognise that a 20 centimetre length could be cut into two ten centimetre lengths, or a nine centimetre and an eleven centimetre length, etc. They should realise that if two litres of water weighs two kilograms then ten litres should weigh ten kilograms.

At Level Two students should be asked to create their own measurement instruments. For example, they might be given a strip of paper and cubes to measure the length of many object thus promoting the creation of a ruler. They may be asked to develop a time measurement device having been shown examples of sandtimers, candle and water clocks, or asked to find the weight of other children given a see-saw.

Effective use of non-standard units will lead students naturally to an understanding of the need for standard units. This can be seen through the effective selection of activities that expose the two main reasons for having standard units. Firstly, there is the difficulty that arises through the use of non-standard units such as hand spans or pencils. That is, that there can be many different hand span or pencil lengths and so the measure of the length of an object will vary according to whose hand span or pencil is being used. Such an understanding can be developed through the use of comparison activities. For example, having the students measure the length of something using their pencils and then comparing results and discussing the reason for the variation in results.

Secondly, if we wish to communicate the result of a measurement to a person in another classroom, city, or country we will all need to be using exactly the same units. This understanding can be assisted by using 'fixed non-standard units' such as the length of a piece of A4 paper as opposed to 'variable non-standard units' such as pencil lengths. Through appropriate examples students will see the need for a single standard system such as the metric system.

Level Three measurement sees a strong focus on **standard units** and specifically the **metric system (SI)**. Appropriate measurement experiences at **Level Two** will have prepared students for the necessity of having standard units so that people all over the world can communicate measurement values and understandings to each other.

Students should be immersed in measurement experiences that are rich in investigation and the use of scales and instruments. As well as gaining a good feel for the size of the metric units, students need to know the names of the units, understand the prefixes and know the symbols for each unit. Teaching students to use scaled instruments such as rulers and protractors effectively, and to read graduated containers is important at this level.

Note that some of the standard units in use today are not actually SI units but can be used in conjunction with SI units. For example, the second is the standard unit of time but the minute and the hour are not actually part of the metric system. However as a society we accept kilometres per hour as a standard unit.

Level Three also sees the start of the use of relationships between length and area, or length and volume, for figures such as rectangles and cuboids. Cuboids are portions of space bounded by rectangles. For example, nearly all packaging boxes are cuboids. So students find equations for the area of a rectangle in terms of the lengths of its sides in whole numbers of units, and the volume of a cuboid in terms of the lengths of its edges in whole numbers of cubes.

Level Four sees a consolidation of understanding and use of the metric system and the further use of scales and measuring devices. Students need to be able use their multiplicative thinking and emerging understanding of decimals to convert between metric units such as grams and kilograms, millimetres and centimetres, most of which they should have met at Level Three. They reinforce their use of formulas to find the areas of rectangles and the volumes of cuboids and extend that to figures such as parallelograms and triangles.

At **Level Five** students should have achieved a level of maturity as measurers that enables them, when given a practical measurement problem, to develop and use a method of solution and discuss the degree of accuracy of the result. Students should apply their understanding of decimals to converting between measures of the same attribute, e.g. 1.276t = 1276kg (t means tonne, 1000kg).

They should be capable of finding ways of measuring the perimeters (circumferences) and areas of circles and using their measurement knowledge to determine the areas of figures that are a composition of known shapes.

In doing so students should connect their understanding of classes of geometric shapes to measurement. For example, a cylinder can be seen as an example of a prism, a solid with constant cross section. The volume of all prisms is found by multiplying the area of its cross section by its height hence the formula $v = \pi r^2 h$.

Teaching Measurement at Level Five:

Faced with real life measurement problems students at Level Five should be able to determine an appropriate way to measure the required attributes, taking into consideration the level of accuracy required for a given purpose. They should be well conversant with the metric system of measurement and should be able to relate measurement units within the system.
Students should be able to show understanding of formulas derived at earlier curriculum levels. They should be able to deduce formulas for areas, volumes, and unit conversions and use them with understanding. They should willingly apply their measurement skills to other areas of the curriculum such as science, health and physical education, and technology, such as using position and orientation concepts (bearings, maps and coordinates) and using spatial properties in design.
Measurement is best understood by measuring. Hence most measurement problems should draw heavily on contexts from the student's physical environment and experience.

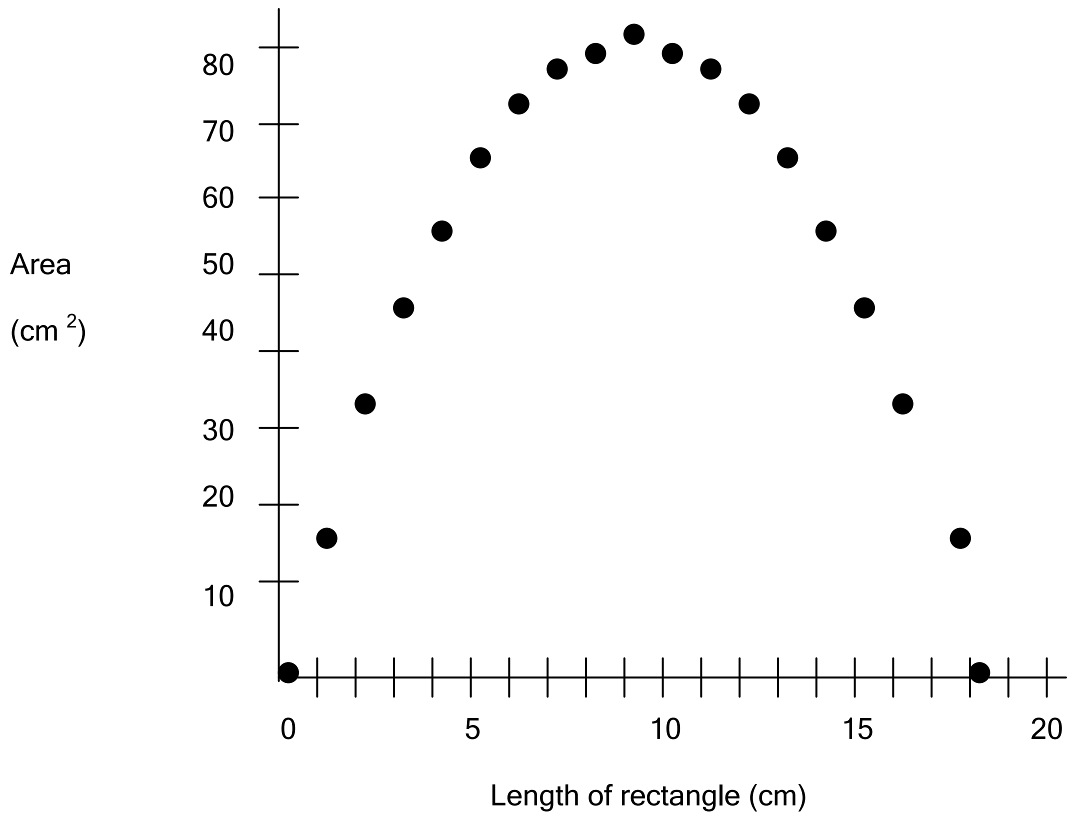
Exemplars of Student Performance:

Exemplar One: Students explore the relationship between area and shape for a rectangle of fixed perimeter.

Students are told that a certain rectangle has a perimeter of a given length. For example, suppose the perimeter is 36 cm. The students are asked what the area is. Many students are convinced that since the perimeter is fixed the area will also be fixed. So they will conclude that the rectangle is a square of side 9cm and therefore has an area of 81 cm^2. Give the students a piece of string and have them tie it to construct a rectangle of perimeter 36 cm. There are infinitely many rectangles of perimeter 36 cm if non-integral side lengths are allowed (e.g. 3.4 x 14.6). Have the students move the string so that they obtain a rectangle of maximum area (a square). Is there a minimum area and what would be the dimensions of the rectangle? To consider these questions further they could construct the following table.

Length	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	cm
Breadth	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	cm
Area	0	17	32	45	56	65	72	77	80	81	80	77	72	65	56	45	32	17	0	cm^2

Having constructed the above table they could draw a graph of the area in relation to the length of the rectangle (see below) and consolidate their concepts regarding the maximum and minimum areas of a rectangle of fixed perimeter.



Students could be asked to solve other related problems, such as:

1. What is the minimum length of electric fence required to enclose a rectangular grazing area of $80m^2$?
2. An existing fence is used as one side of a rectangular grazing area. What is the maximum area that can be enclosed by a 64m length of electric fencing if the existing fence is used?
3. What shape maximises area for a given perimeter, e.g. 81m? (circle)

Exemplar Two: Students explore the relationship between the perimeter (circumference) of a circle and its diameter. They determine an approximate value of π and express the relationship as a formula.

Students need to explore this relationship using solid objects and string. It is important that they have several shapes with circular cross-sections- about six is sufficient. It is also important that the shapes are of differing diameters because many students intuitively believe that the relationship holds for a specific circle and varies according to the length of the diameter.

A good introduction is to ask students to look at a large circle (such as a hula-hoop) and estimate the number of diameters that could be wrapped around the circumference. Record their estimates as a later comparison once π is found.
The students then wrap string around the circumference of each object to measure the length of the circumference. A mark can be made on the string to help with the measurement. For the smaller objects it might be more accurate to wrap the string around several times (e.g. ten times) and calculate the measurement from that. If several groups are dong the measuring they can average their results to get a more accurate measurement. It is important for relationship spotting that, as the students measure, they record both the length of the diameter and the circumference of each circle. A table such as the one below can then be filled in.

Object	**Average length of diameter (*d*)**	**Average length of circumference (*c*)**
Jam jar	7.4 cm	23.2 cm
Cotton reel	3.7 cm	11.6 cm
Preserving jar	11.5 cm	35.7 cm
Large saucepan	37.6 cm	118.0 cm
Poster cylinder	12.0 cm	38.0 cm
Sellotape reel	13.1 cm	41.1 cm

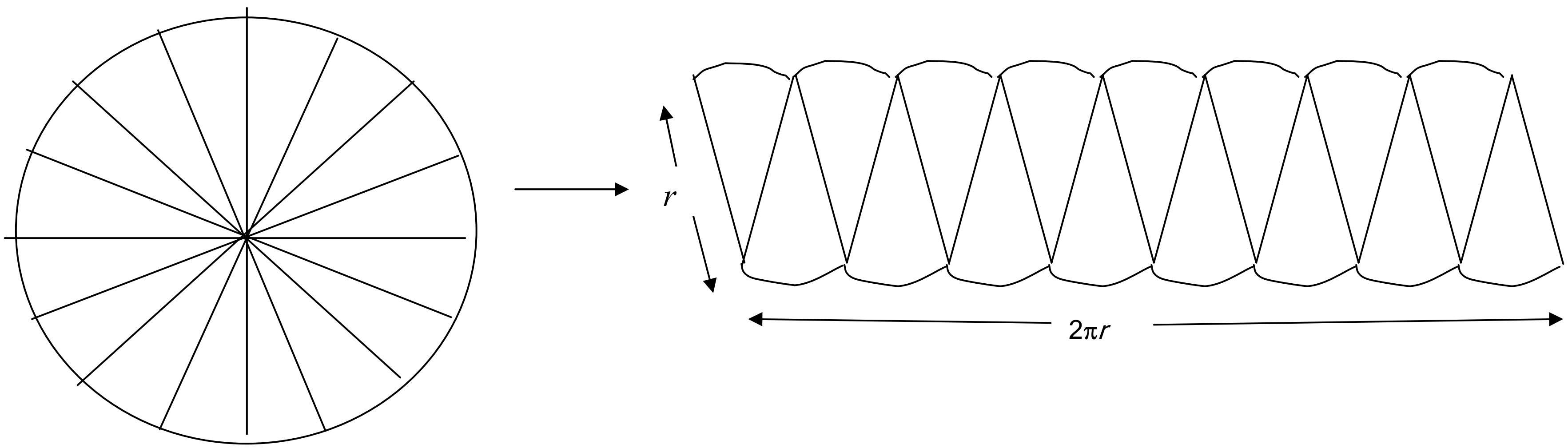
The students are asked to examine the table for a relationship between the diameters and circumferences. Rounding the numbers to the nearest whole number can help students recognise that the length of the circumference (*c*) is slightly over three times the length of the diameter (*d*). The students then use calculators to find *c* divided by *d*. The table below shows the results.

Object	**Average length of diameter (*d*)**	**Average length of circumference (*c*)**	***c* ÷ *d***
Jam jar	7.4 cm	23.2 cm	3.14
Cotton reel	3.7 cm	11.6 cm	3.14
Preserving jar	11.5 cm	35.7 cm	3.13
Large saucepan	37.6 cm	118.0 cm	3.14
Poster cylinder	12.0 cm	38.0 cm	3.16
Sellotape reel	13.1 cm	41.1 cm	3.14
		Average	3.1417

From this the students can see that there is a fixed ratio of circumference to diameter and it is independent of the size of the circle. The name given to this ratio (pi) and its symbol (π) can then be discussed.
Note: π is an irrational number and therefore cannot be exactly expressed as a ratio of integers (a fraction), or as a repeating or terminating decimal. Consequently its decimal form has no pattern of repetition. This activity gives the opportunity to discuss the idea of such numbers with students. To twenty decimal places, π has the representation: π = 3.14159265358979323846 (to 20 dp)

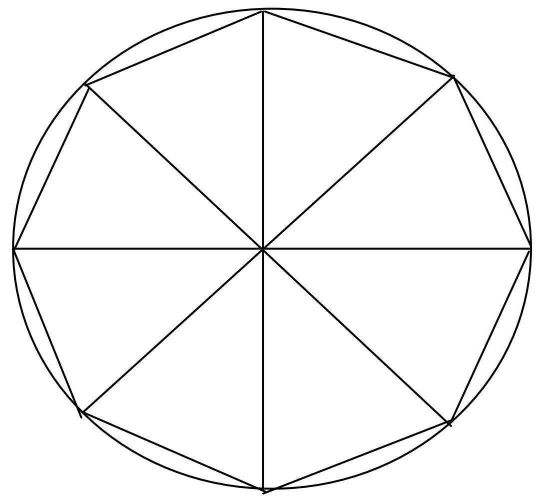
Exemplar Three: Students use approximation techniques and intuitive limit concepts to find the area of a circle of radius r. They express the area as a function of r.

Students need to have explored and discovered the formula for the circumference of a circle, namely $c = \pi d$ (or $c = 2\pi r$) before exploring the area of a circle. The students are provided with a circle divided into equal sectors. About 16 sectors is ideal. The students colour the arc of one semi-circle one colour (e.g. blue) and the arc of the other semi-circle another colour (e.g. red). The students then cut out the sectors and arrange them into a shape that closely resembles a parallelogram, with all the red edges on one side of the parallelogram and all the blue edges on the opposite side.



Students could cut up different circles into a greater number of equal parts, e.g. 32 parts, 64 parts, etc., and rearrange these parts to form parallelograms. From this the students can see that the greater the number of sectors into which the circle is divided, the more closely the shape resembles a rectangle with length πr (half the circumference) and breadth r. Hence the circle has area πr^2.

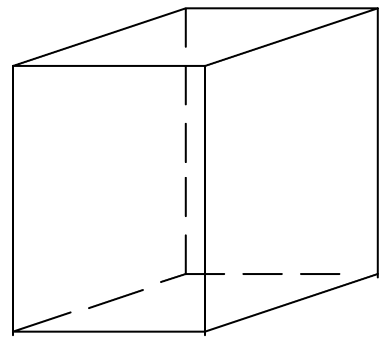
As an extension of this exploration, students could consider dividing a circle into triangles. For example, suppose we divided a circle into eight triangles and considered the area of those triangles.



The area of each triangle is one half of the length of the base (b) times the height (h). So the area of the octagon is 4bh. Suppose we created a 100-gon inside the circle. Then the area of the 100-gon would be 50bh. The perimeter of the 100-gon would be 100b and would be very close to *c* (the circumference). The height of the triangles (*h*) would be very close to the radius of the circle (r). So if the area of the circle is A, then $A \approx 50bh \approx Cr \div 2 = 2\pi r^2 \div 2 = \pi r^2$

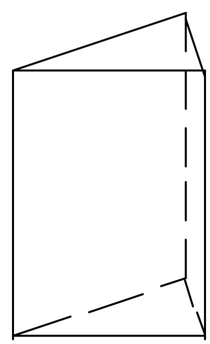
Exemplar 4: Students explore the volumes of prisms using practical techniques. They discover a formula for the volume of a prism. Begin by reviewing students' knowledge of the volume of a **cuboid**, which is a solid figure bounded by six rectangular faces. Have students form cuboids from unit cubes to confirm (or discover) that the volume of a cuboid is found by multiplying the base area by the height. Note that a **prism** is a polyhedron with two congruent and parallel faces (the bases), whose remaining faces (lateral faces) are parallelograms. Hence a prism can be skewed (on a lean, so to speak) so long as its bases are congruent and parallel. At this level we will deal only with **right prisms**, which are prisms where the lateral faces are perpendicular to the bases – such as a **cube**)

Cuboid



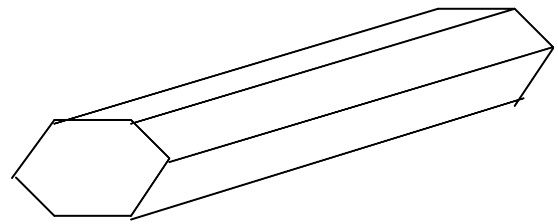
Have students consider the volume of a **right triangular prism**. This is a prism whose parallel faces are triangles and whose lateral faces are all **perpendicular** to the parallel faces. Begin with a right triangular prism whose triangular bases **are right-angled triangles**. The volume of such a right triangular prism can be found by considering the right triangular prism as one half of a cuboid. Students could model a cuboid out of some substance, such as clay or plasticine, and show that it can be cut in half to give two right triangular prisms. Hence the volume of the right triangular prism is one half of the volume of the cuboid and therefore can be found by multiplying the area of the triangle by the length of a rectangular side. Students could reasonably generalise this to show that the volume of any right triangular prism is the product of the area of the base triangle and the length of a rectangular side.

Right triangular prism

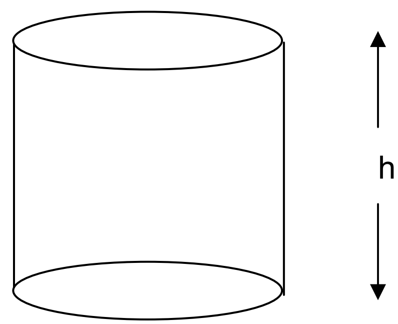


If the base of a right prism is some polygon other than a triangle, students could consider that all polygons can be subdivided into triangles and hence the **volume of any right prism** is the product of the area of the base and the length of a rectangular side.

Right hexagonal prism



If this thinking is extended to a **right cylinder** it should be apparent that the volume of the cylinder is the product of the area of the base and the height of the cylinder. So if the radius of the circular base is r and the height of the cylinder is h then the volume of the cylinder is $\pi r^2 h$.



A Practical Confirmation of the Above:

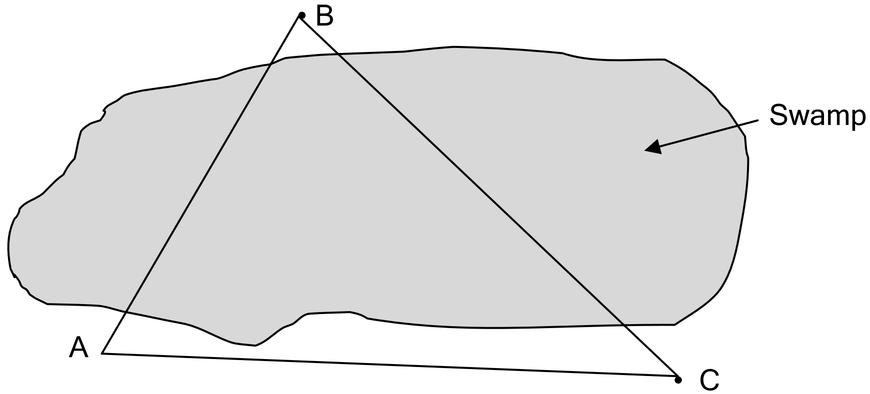
To have the students confirm that the volume formulae derived above are correct, a good (if slightly wet) approach is to have the students carefully immerse shapes of the type considered above into a full pan of water and collect and measure the overflow. The students draw up a table with the dimensions of the shapes in centimetres. It is best to have a variety of sizes of the same type of shape. Measuring the overflow in millilitres will confirm for the students the formulae derived above.

As an extension activity, students could use this method to find the volume of a right circular cone. Have the students hypothesise a possible formula first. It is likely that most students will propose the volume of the cone to be one-half the volume of a cylinder of the same base area and height as the cone. Using the overflow method will give them the correct result of one-third the volume of the cylinder, i.e. $^{1}/_{3}\, \pi\, r^{2} h$

Exemplar 5: Students extend their knowledge of the areas of polygons to include the areas of trapeziums.
Students might need to be reminded of the formula for the area of a rectangle. They might also need to rethink how they discovered the formula for the area of a parallelogram by transforming a parallelogram into a rectangle. This activity would be best started on squared paper with the length of the parallel sides of the trapezium being an even number of squares.

Because the trapezium is drawn on squared paper it is easy for the students to see that the trapezium can be considered as a 4 cm by 6 cm rectangle and two right triangles. They know how to find the areas of the right triangles and can therefore see that the area of the trapezium is 54 cm^2. However a transformation can help the students to develop a more general formula for the area of the trapezium. It is a good idea to shade each edge of the trapezium a different colour.
Students are asked how they might transform the trapezium into a shape they know. By cutting off each triangular shape at its midpoint and rotating it they can see the shape has changed from a trapezium to a rectangle (outlined in grey) and the lengths of the parallel sides which were 4 and 14 are now both 9. Why are they both 9? They are both 9 because that is half the sum of 4 and 14 (or in other words the average of the two numbers) and what we have done in creating a rectangle is to average the two parallel sides. Hence the area of the trapezium is best seen as half the sum of the two parallel sides multiplied by the distance between them, the height of the trapezium.

Exemplar 6: Students use scale drawing to find the distance to an inaccessible object.
Students are required to find the distance between two points A and B. They cannot measure the distance because an imaginary swamp lies between the two points. They discuss the possibilities. One simple method is to mark a third point, C, that they can access. They find the angles at A and C using a protractor or a simple theodolite. They also measure the length of the line segment AC.



They then draw to scale the map as shown and scale off the length of the line segment AB.

Exemplar 7: Students create a map of the school (or part of the school) by plane table. They explore angles, distances, and areas.
For this activity students need a desk, a piece of A3 paper (cartridge paper would be good) and a long ruler. The paper is fixed to the desk using blutak™ or similar. Two points on the ground are selected. The points (A and B) need to be such that most of the defining landmarks/features of the map can be seen from them. The further apart they are the more accurate the map will be. An appropriate scale is chosen, e.g. 1cm = 1m.
The table is placed over the point A and the line to B is drawn to scale on the paper, with the paper line AB in the same direction as the line AB on the ground. Lines to defining landmarks of the map are drawn on the paper using the ruler in the direction of those landmarks. Each line needs to be labelled as to what point it has been drawn to.
The table is then placed over the point B with the paper line BA in the direction of the ground line BA, and the lines to the same landmarks are drawn from point B on the paper. A landmark positions on the map where the two lines to it intersect. When the landmarks have been established, the drawing lines can be rubbed out, and the students can complete the map. Students could measure some distances on the ground to see how accurate the map is.
The map could then be used for things such as finding the area of the grassed region by dividing it into triangles.

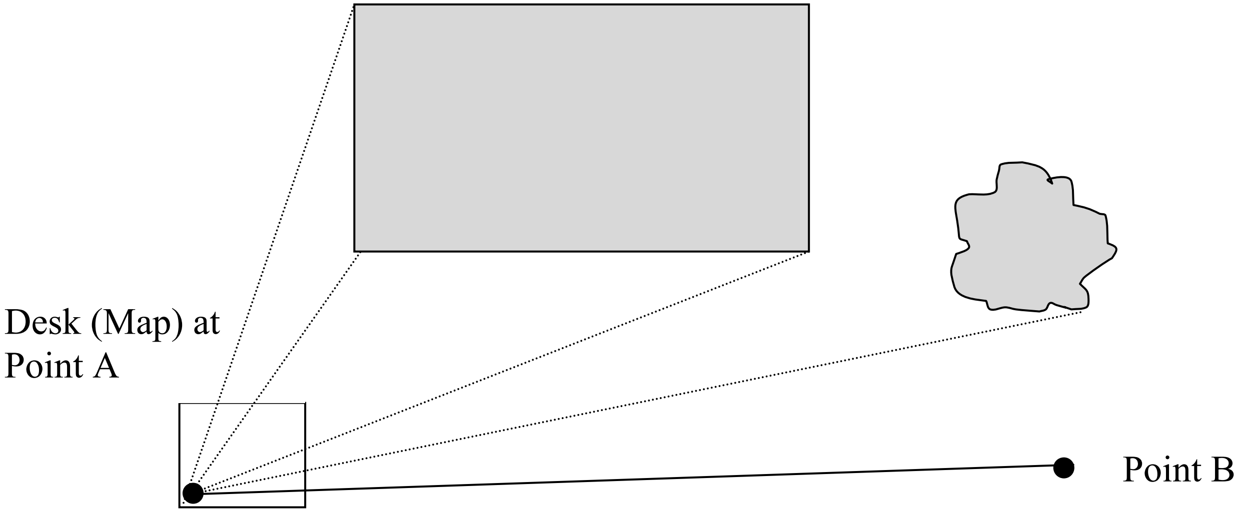


Illustration of plane table situated at point A with lines drawn towards significant points on buildings.
(Significantly not to scale)

Exemplar 8: Students follow a trail by measuring angles and distance from a map.
Students are given a map of a local area such as the school or a reserve. Marked on the map is a sequence of points A, B, C... It is best if other topography, such as buildings, is not on the map. Students work out the angles and distances off the map using protractors and rulers (scale). They determine the direction of the next point using either a large protractor or a compass. If students use a magnetic compass they will need to allow for local magnetic variation, which can be checked on the internet. Alternatively they could use an enlarged protractor mounted on cardboard. They travel the distance to the point to locate a small marker using a trundle wheel or tape measure to determine distance. Alternatively students can pace out the distance.* If each marker contains a letter, the completed journey can give a message or a key word that may direct the students to the location of hidden treasure.
* Using 30 metre tape with points marked on a sealed area, each student can determine the length of their pace and use that to measure distance.

Exemplar 9: Students find an expression for the surface area of a cylinder of radius r and height h.
Students are supplied with a cylindrical tin such as a 420gm baked bean tin. They already know that the perimeter of a circle of radius r is $2\pi r$ and that the area of a circle of radius r is πr^2. They cut a piece of paper to size to wrap around the tin. They see that the rectangular piece of paper has length $2\pi r$ and width h, and therefore has area $2\pi rh$.
Consequently they deduce that the surface area of the cylinder is $2\pi rh + 2\pi r^2$.
Extension: If they already know that the volume of a cylinder is $\pi r^2 h$, an interesting project would be to see if a cylinder of the same volume but a smaller surface area could have been made, thereby saving the amount of metal used. This is a good problem to solve using a spreadsheet.

Useful resources

Figure It Out:
Measurement, Year 7-8, Book 2 (Level 4+): Speed (p.1, 24), Length (pp.2-3, 6-7, 16-17), Mass (pp.4-5), Area (p.10,12-13, 18-19), Volume (pp.8-9,14-15, 20-21), Time (p.11, 22-23).

Numeracy Project Book 9: Teaching Number through Measurement, Geometry, Algebra, and Statistics, pages 3-15.
nzmaths.co.nz units (This website is sponsored by the Ministry of Education)
http://www.nzmaths.co.nz/node/416 (Time: Round the Track)
http://www.nzmaths.co.nz/node/214 (Shape, Transformation and Measurement: How high and Other Problems)
http://www.nzmaths.co.nz/node/417 (Length: Around we Go)
http://www.nzmaths.co.nz/node/425 (Area: Fences and Posts)
http://www.nzmaths.co.nz/node/353 (Transformation, Area, Volume and Number: Scale Factors for Areas and Volumes)
http://www.nzmaths.co.nz/node/792 (Volume: Measurement Investigations I)
http://www.nzmaths.co.nz/node/794 Volume: Measurement Investigations II)

Digital Learning Objects (These are accessed through the Ministry of Education Digi-Store and are the result of a collaborative project run by The Learning Federation, Australia)
http://www.nzmaths.co.nz/learningobjects/315/5

Other Website links:
http://nlvm.usu.edu/en/nav/category_g_3_t_4.html