Mathematics in the New Zealand Curriculum
Second Tier

Strand: Number and Algebra | Thread: Patterns and Relationships | Level: Three

Achievement Objectives:

- Generalise the properties of addition and subtraction with whole numbers.
- Connect members of sequential patterns with their ordinal position and use tables, graphs and diagrams to find relationships between successive elements of number and spatial patterns.

Exemplars of student performance:

Exemplar One:

Quentin (student) solves equations involving addition and subtraction with unknowns on either side of the equals sign. He is able to solve the equations relationally. This means that he applies invariant number properties rather than calculates the total value for each side of the equation. He understands the equals sign as a statement of balance that preserves under any transformation of the terms in the equation.

To solve:

a. 98 + 46 = □ + 44: Quentin notices that 44 is two less than 46. He knows that □ must be two more than 98 (100) for the equation to hold.

b. 86 - □ = 83 - 47: Quentin realizes that both sides of the equation must have the same difference. Since 86 is three more than 83 then □ must be three more than 47 (50).

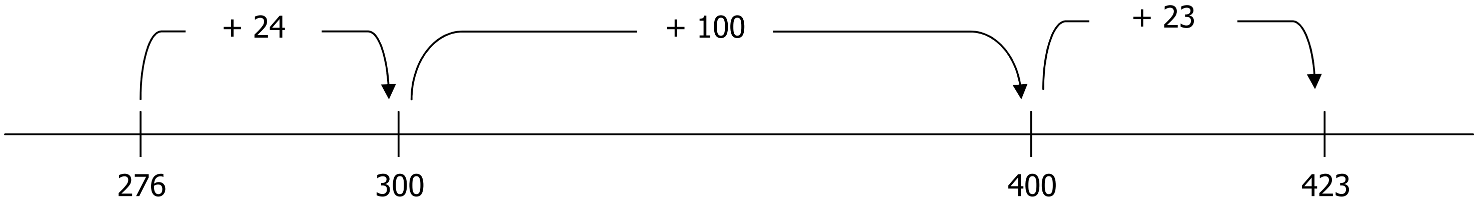
c. 635 - 77 = 600 - □: Quentin recognises that 600 is 35 less than 635 so to preserve the equality □ must be 35 less than 77 (42).

d. 8 × □ = 7 × 4: Quentin combines the commutative property (8 × 7 = 7 × 8) with doubling and halving to conclude that □ = 14 (since 8 × 7 = 4 × 14).

Quentin's responses show achievement at level three because he thinks relationally about equations involving addition and subtraction and simple multiplication equations. This shows that he understands key properties of addition and subtraction such as the commutative property, e.g. 56 + 87 = 87 + 56, compensation, e.g. 98 + 76 = 100 + 74, and equal differences, e.g. 81 – 58 = 83 – 60.

Exemplar Two:

Aimee (student) solves the subtraction problem, 423 - 276 = □ as 276 + □ = 423 using this empty number line.

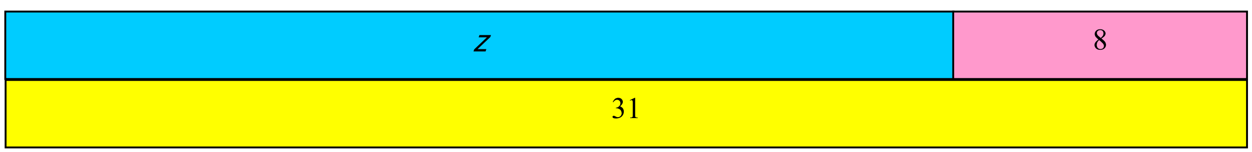


She says the answer is 24 + 100 + 23 = 147 and checks the calculation by confirming the ones digit is correct, using 13 – 6 = 7 .

> Aimee's work shows achievement at level three because she understands addition and subtraction as inverse operations, that is, one operation undoes the other. So for any subtraction problem c – b = □, where c and b are numbers, there must be a two matching addition problems, □ + b = c and b + □ = c. In this case 423 – 276 = □ so 276 + □ = 423.

Exemplar Three:

Greg has this diagram, and solves for *z*:

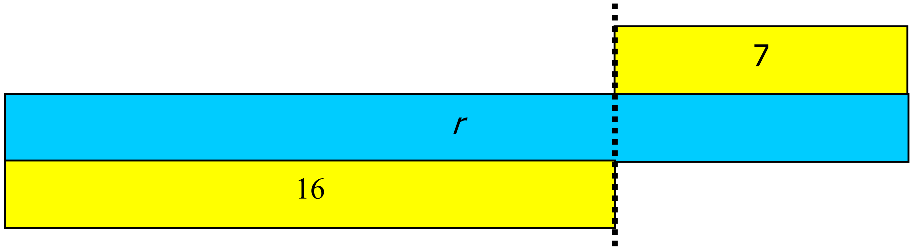


He recognizes that z equals 31 – 8 = 23. Greg can write the problem algebraically as:

$z + 8 = 31$

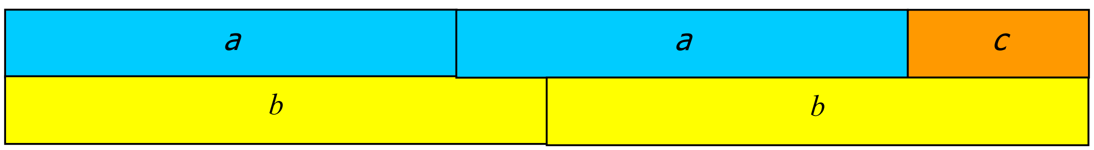
so $z = 23$

Similarly Greg can solve for r in this problem:

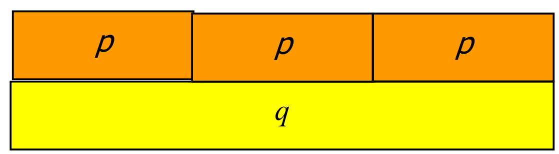


He recognizes that $r = 16 + 7$ and can write the problem algebraically as:
$r - 7 = 16$
so $r = 23$

Kyla uses a set of strips and establishes relationships between the lengths.
For example, she writes $a + a + c = b + b$ for this relationship (Note this can be written as $2a + c = 2b$), and realizes that this means $b = a + \frac{1}{2}c$ (b equals a plus one half of c).

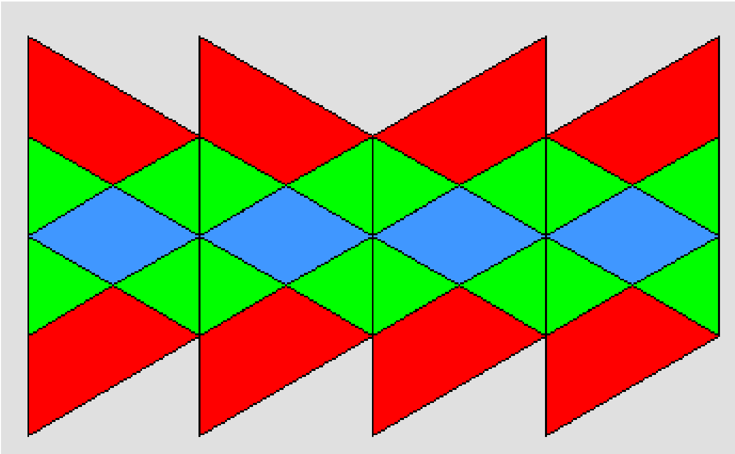


For this relationship she writes $3p = q$ and $p = \frac{1}{3}q$.



Greg and Kyla's work exemplifies Level Three because they are able to recognize and describe additive and simple multiplicative relationships. In Kyla's case she is able to describe those relationships despite the fact that the strips have no specific measurement assigned to them. The ability to handle objects as variables indicates acceptance of "lack of closure."

Exemplar Four:
Sonita, Villiame and Matiu work out the number of triangles and trapezia (red shapes) needed to match with 10 rhombi (blue shapes) in this paving pattern.



a. Sonita notices that for every diamond that is added there are two trapezia and four triangles added. So she creates a table like this and continues the pattern by adding.

Number of rhombi (blue)	1	2	3	4	5	6	7	8	9	10
Number of triangles (green)	4	8	12	16	20	…				40
Number of trapezia (red)	2	4	6	8	10	…				20

+ 4

+ 2

Villiame sees the same relationships as Sonita but uses multiplication instead of addition to find the values he needs. To find the number of triangles he uses 10 × 4 = 40 (since there are 4 triangles for every rhombus) and to find the number of trapezia he uses 10 × 2 = 20 (since there are 2 trapezia for every rhombus).

Matiu uses a computer spreadsheet to find the missing values. He uses the fill down function to extend the patterns down, e.g. by clicking on the bottom right corner of the block and dragging it 8 cells down.

C2 fx 2

	A	B	C
1	Rhombi	Triangles	Trapezia
2	1	4	2
3	2	8	4
4			

→

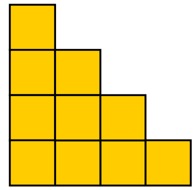
A11 fx 10

	A	B	C
1	Rhombi	Triangles	Trapezia
2	1	4	2
3	2	8	4
4	3	12	6
5	4	16	8
6	5	20	10
7	6	24	12
8	7	28	14
9	8	32	16
10	9	36	18
11	10	40	20

Sonita and Matiu all show level three achievement as they use additive strategies to find missing values in a sequential pattern. They are all able to create general rules for the relationships. Both Sonita and Matiu use recursive rules (add two or four to get the next numbers in the pattern). Villiame uses a direct rule (multiply by two or four). Matiu recognizes the power of technology to solve sequential patterning problems. Note that it is a requirement of Level 3 that students have the capacity to create and apply additive recursive rules, while direct multiplicative rules for simple linear relations are a level four requirement. So Villiame's strategy exemplifies level four achievement.

Exemplar Five:

Wikitoria (student) is solving this problem:
How many square tiles would be in this
Staircase pattern if it was ten tiles high?
She begins by building the staircase as follows:



Wikitoria notices that she needed to add one more tile than the bottom layer each time. She created a table rather than building up the staircase anymore.

Height	1	2	3	4	5	6	7	8	9	10
Number of Tile (Total)	1	3	6	10	15	21	28	36	45	55

+ 2   + 3   + 4   + 5   + 10

> Wikitoria's strategy exemplifies Level Three because she used an additive general rule to find a missing value in a sequential pattern. Note that finding a direct rule for non-linear patterns like this, i.e. connecting height with tiles, is not expected until Level Five.

Exemplar Six:

Taylor investigates this pattern of equations:

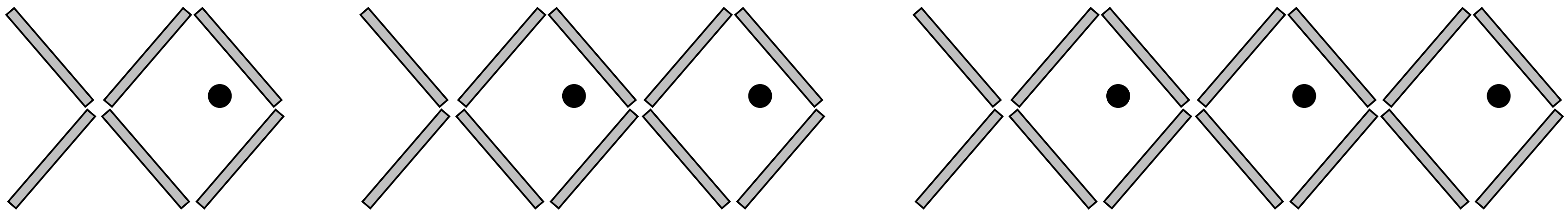
$$20 - 10 + 5 = 20 + 5 - 10$$
$$8 - 4 + 2 = 8 + 2 - 4$$
$$64 - 32 + 16 = 64 + 16 - 32$$

She records other equations that she believes fit the same pattern, e.g. 100 - 50 + 25 = 100 + 25 - 50. Taylor describes why she thinks each equation fits the pattern in this way, "Whatever number you choose for the first number you take half of it away then add one quarter of it. That gives the left side numbers. The numbers on the right are the same except you do the addition before the subtraction." Asked if the starting numbers are special she replies, "It is easier if you choose numbers that are answers to the four times table."

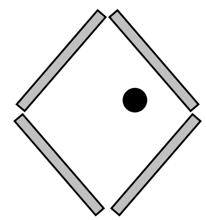
Taylor's achievement exemplifies Level Three because she recognizes relationships common to all three equations across the equals sign. These relationships are additive or simple multiplicative, e.g. halving and multiples of four. Using these relationships, she is able to create other equations that fit the pattern. She recognizes that the order of carrying out the addition and subtraction operations does not alter the equality.

Important teaching ideas (working at):
The focus of algebra at Level Three is generalization of the additive and early muliplicative number properties, application of these properties to relations, and emerging competence in expressing these properties using the language of variables. For students working at level three, generalisation involves these progressions (Mason, 2003):

1. **Gazing at the whole**
For example, a student sees this growing pattern of fish made from matchsticks. The pattern captures their attention.

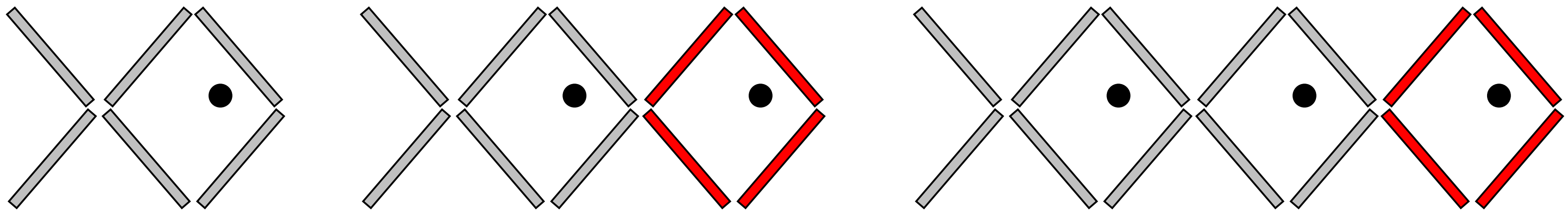


2. **Discerning details**
The student recognizes that the pattern is composed of a repeating element.
He or she notices that the first face has two extra matchsticks.



3. Recognising relationships

The student notices that the next pattern in the sequence has four more matchsticks than the previous pattern:



4. Perceiving properties

The student sees that an additive relationship exists between the number of matchsticks needed for a given number of fish and the number needed for the next number of fish. Since four matchsticks are added each time the relationship is linear and involves an "add four" characteristic.
Since the first face took six matchsticks, two extras must be added to compensate:

Number of faces	1	2	3	4		10		100
Number of matchsticks	6	10	14	18		42		402

+4 +4 +4

From their work on number strategies, students will have access to additive and early multiplicative properties without consciously thinking about these properties as generalizations. Require students to interpret the number strategies of others, leading to finding unknowns in equations without calculating, then expressing the strategies as generalizations. Introducing letter symbols can be appropriate but word rules and diagrams are equally valid descriptions. The focus is on connecting the generalization with the symbols. Note that students need to interpret the "equals sign", =, as a statement of relational balance rather than as an indicator that the answer follows, as in equations like $4 + 5 = \square$ which students are most familiar with. Examples illustrating the key properties of the multiplicative field follow:

Additive Property	Given strategy	Missing unknown	Algebraic substitution (**Extension only**)	Algebraic Generalisation (**Teacher only**)
Commutative Addition	S solves 36 + 467 = □ as 467 + 36 = □. How would S solve 18 + 4 563 = □?	What number does Δ represent in each equation? 78 + 56 = 56 + Δ Δ + 97 = 97 + 79	What would go in each □ to make each equation balance? 93 + k = □ + 93 □ + 64 = 64 + d	$a + b = b + a$
Associative Addition	K solves 36 + 59 + 64 = □ as 36 + 64 + 59 = □ How would K solve 89 + 49 + 51 = □?	What number does Δ represent in each equation? 47 + 58 + 37 = 58 + Δ + 37 Δ + 61 + 39 = 100 + 54	What would go in each □ to make each equation balance? 1000 + z = 450 + □ + 550 □ + 28 + 22 = 50 + g	$(a + b) + c = a + (b + c)$ Note that parentheses indicate to “do first”.

Multiplicative Property	Given strategy	Missing unknown	Algebraic substitution (**Extension Only**)	Algebraic Generalisation (**Teacher Only**)
Addition and subtraction as inverses	T solves 48 + 77 = 125 so he knows 125 – 48 = 77 and 125 – 77 = 48. What does T know from 345 – 187 = 158?	What number does Δ represent in each equation? 67 + 74 = 141 so 141 - Δ = 67 83 – 45 = 38 so 38 + Δ = 83	What would go in each □ to make each equation true? 46 + h = 92 so 92 – 46 = □ w - 81 = 33 so 81 + 33 = □	$a + b = c$ so $c - b = a$ and $c - a = b$ $x - y = z$ *so* $x - z = y$ and $y + z = x$
Equal Additions Property of Subtraction	N solves 96 - 38 = □ as 98 – 40 = □. How would N solve 125 - 49 = □?	What number does Δ represent in each equation? 103 - 77 = Δ - 80 83 - Δ = 81 - 68	What would go in each □ to make each equation balance? 231 - f = 234 - □ □ - 98 = 226 - 100	$a - b = (a + n) – (b + n)$ $= (a - n) – (b - n)$
Commutative Property of Multiplication	Y solves 48 × 3 = □ as 3 × 48 = □. How would Y solve 333 × 4 = □?	What number does Δ represent in each equation? 26 × 83 = 83 × Δ Δ × 76 = 76 × 34	What would go in each □ to make each equation balance? 76 × k = □ × 76 □ × 84 = 53 × d	$a \times b = b \times a$ *or* $ab = ba$
Distributive Property of Multiplication	U solves 12 × 6 = □ as (10 × 6) + (2 × 6) = □. How would T solve 23 × 8 = □?	What number does Δ represent in each equation? 9 × 11 = (9 × Δ) + (9 × 1) Δ × 46 = (40 × 7) + (6 × 7)	What would go in each □ to make each equation balance? 9 × □ = (10 × d) - (1 × d) 97 × q = (97 × 5) + (97×□)	$a \times b = (a \times c) + (a \times d)$. or $ab = ac + ad$, where $c + d = b$ or $a \times b = (a \times c) - (a \times d)$. or $ab = ac - ad$, where $c - d = b$

At Level Three sequential and repeating patterns are used to develop ideas about relations. Relations are sets of ordered pairs, e.g.{(2,7), (3,11), (4, 13) …}. The first value in an ordered pair represents a possible value of the independent variable, i.e. the variable you are free to manipulate. The second value represents the corresponding value of the dependent variable, i.e. the variable for which its value is determined by the value of the independent variable. In the matchstick pattern above, the number of fish was the independent variable and the number of matchsticks was the dependent variable. Graph relations on a number plane with the values of the independent variable on the x-axis (across) and the values of the dependent variable on the y-axis (vertical). In mathematics, we tend to be more interested in relations that show a graphical pattern than those that don't.
At Level Four students need to look for both recursive (repeating) and direct relationships in sets of ordered pairs. Recursive relationships involve looking at what happens to a term in a sequence to get the next term. A direct relationship links the values of the independent variable to those of the dependent variable. For students at Level Three direct relationships are required to be either additive, e.g. take away 5, or simple multiplicative, e.g. times 3. This should not prohibit students from being exposed to more complex relations.

Tables and graphs are important representations for looking at relationships. Use computer technology whenever possible to speed up the creation of these representations and allow students to see how changes in table values affect the corresponding graph.
The examples below use the fishes matchstick pattern as a context.

Set up the table columns as below. Use the fill down function to generate the x-values. Click on the bottom right corner of the blocked area and scroll down the column with the left button down. This fill the sequence of counting numbers into column A.

Type in the first y-value, 6, in cell B2. In cell B3 type the function =B2+4 then hit return. This will take the value in B2 add two to it, and put the answer in B3.

	A	B
1	Faces	Matchsticks
2	1	
3	2	
4		

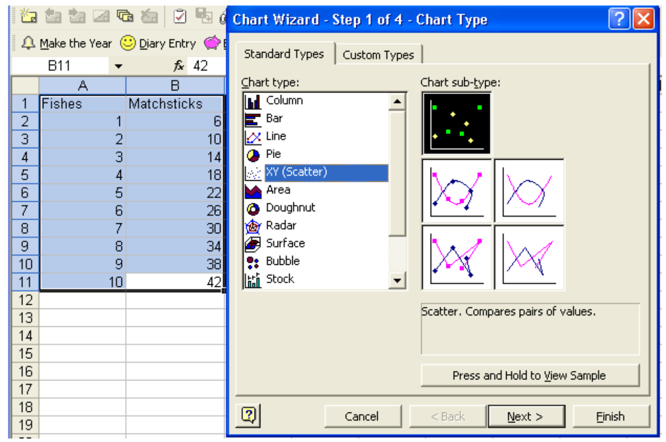
By clicking on the bottom right corner of cell B3 and scrolling down with the left mouse key on will fill the "add four to the cell value above" pattern, as shown.

B3 f_x =B2+4

	A	B	C
1	Faces	Matchsticks	
2	1	6	
3	2	10	
4			

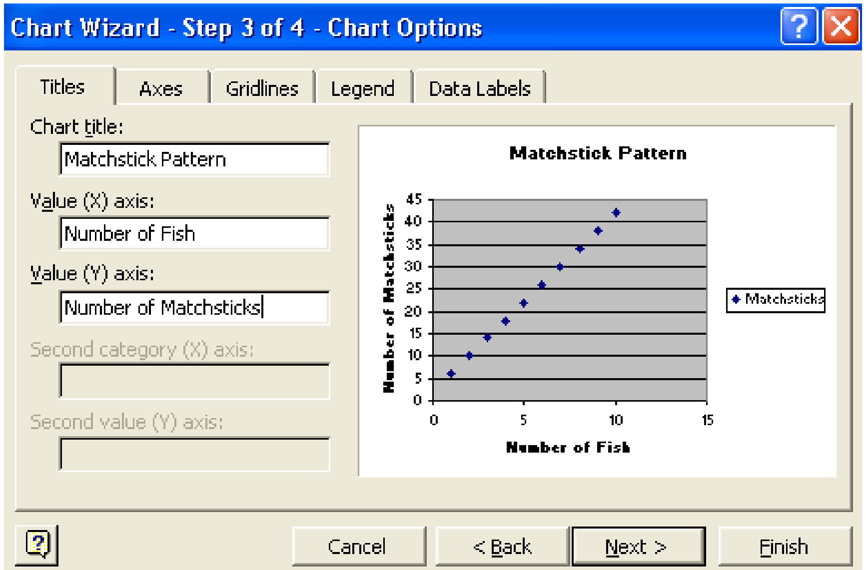
Block the whole table using your mouse, holding down the left button. Click the graphing icon (bar graph) in the toolbar. Choose "Scatterplot" to graph the relation.

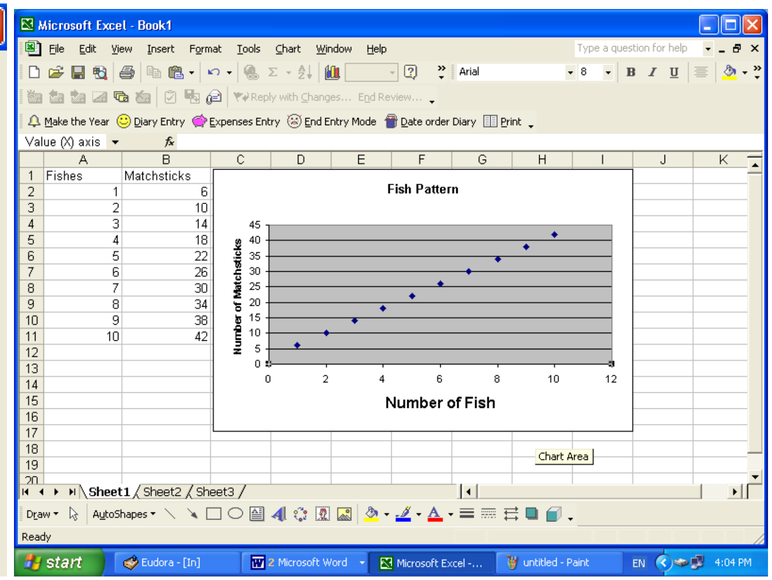
	A	B
1	Fishes	Matchsticks
2	1	6
3	2	10
4	3	14
5	4	18
6	5	22
7	6	26
8	7	30
9	8	34
10	9	38
11	10	42



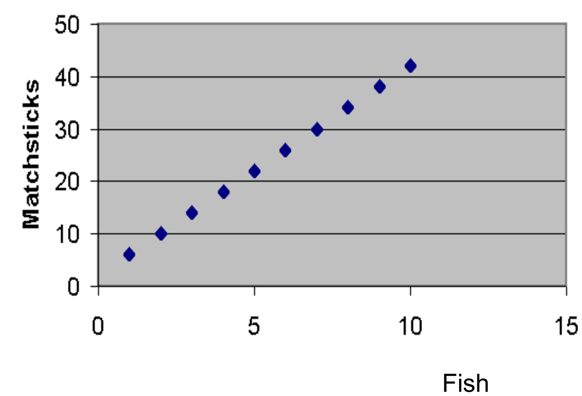
Click next on the Chart Wizard until this screen appears.
Type in the title, and x and y values.
Click off the legend.

Continue clicking next in the Wizard. You will get a choice of placing the graph into the existing sheet (as shown) or creating it as a separate sheet. For presentation a separate sheet is best. For continued interaction with the table choose the default.

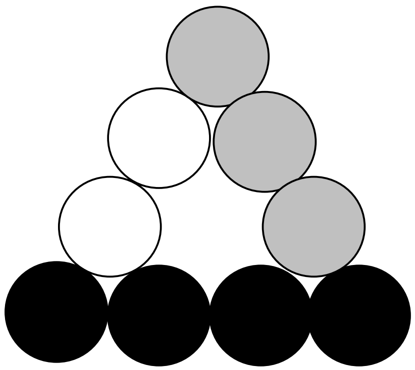




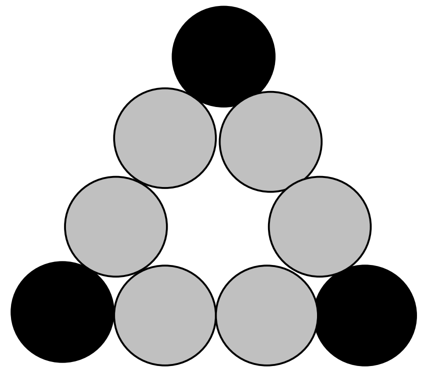
If the graph is embedded within the table sheet the values in the table can be changed and the graph will alter accordingly. It is important for students to realize that this pattern is linear. The ordered pairs lie on a line.



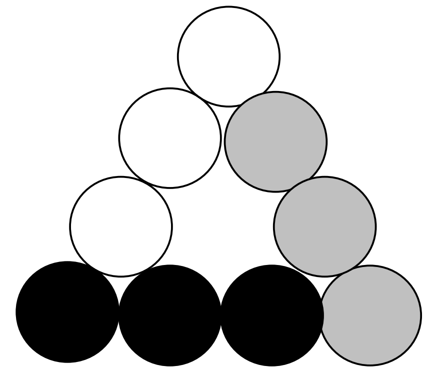
The three main contexts to encounter relations in are geometric, numeric and statistical (measurement). Contexts can be a combination of these. Geometric pattern has the advantage of allowing students to take advantage of both spatial and numeric reasoning. For example, consider this triangular border pattern. Students attendance to spatial grouping of the circles affects how they answer the question, "How many circles would be in a border pattern 10 circles across the bottom?"



Iria see this…



Leigh sees this…



Paora sees this…

Their calculations for the pattern 10 circles across are:

10 + 9 + 8 = 27

8 + 8 + 8 + 3 = 27

9 + 9 + 9 = 27

It is important that discussion about the rules students create focus on:

1. Structural thinking about where the numbers came from,
   e.g. "What do you see in your head that gives you 8 + 8 + 8 + 3?" "What circles are counted with each lot of nine?"
2. Extension of the rule to other examples,
   e.g. "If I said the pattern had 100 circles along the bottom, what would you do then?"
3. Generalising the rule so it applies to any member of the pattern,
   "So if I give you the number of circles along the bottom, what do you do?"
4. Expressing the rule (this may lead to symbols or be left as words),
   "How could we say your rule as simply as possible?" "How could we write it down?"

At higher levels students explore the structural equivalence of different rules for the same pattern. See Level Four for details.

Numeric patterns have the advantage of minimizing noise from the problem contexts. Equation patterns and function machines provide useful vehicles for relational thinking.

Below are some examples of equation patterns:

1 + 2 = 2 + 1	1 + 4 = 2 + 3	10 - 5 = 9 – 4	1 + 2 – 3 = 0
2 + 3 = 3 + 2	2 + 5 = 3 + 4	12 - 6 = 11 – 5	2 + 4 – 6 = 0
3 + 4 = 4 + 3	3 + 6 = 4 + 5	14 - 7 = 13 – 6	3 + 6 – 9 = 0

Ask the students to:

1. Write the next two equations in the pattern.
2. Write an equation that is "a long way down" in the pattern.
3. Write a rule for all equations in the pattern.

For example, in the pattern beginning as 1 + 4 = 2 + 3, students might write 4 + 7 = 5 + 6 and 5 + 8 = 6 + 7 as the next equations. In doing so, they notice that the addends increase by one. To write an equation further down they may choose any whole number to start the equation, e.g. 16. They then need to find the relationships across the equals sign to complete their equation:

+ 2

+ 3

$$16 + 19 = 17 + 18$$

+ 1

Expressing a general rule for the equation set requires the use of some symbol to represent the first addend in the equation. We develop this idea at later levels. Suppose *a* represents the first addend then the general rule can be written as:

$$a + (a + 3) = (a + 1) + (a + 2)$$

Function machine contexts involve the creation of ordered pairs. Enter numbers into a hypothetical machine that performs a consistent transformation on them to produce output numbers. Ask students to find the algorithm (rule) that the machine is using. It is important not to enter the numbers in sequence if direct rules are to be encouraged. Below are some examples:

In	Out
7	11
4	8
12	16
2	6
n	*n*+4

In	Out
8	2
13	7
20	14
6	0
n	*n*-6

In	Out
6	12
3	6
10	20
8	16
n	2*n*

In	Out
8	4
14	7
3	1.5
10	5
n	½*n*

In	Out
3	9
9	27
5	15
11	33
n	3*n*

In	Out
3	8
6	14
0	2
10	22
n	2*n*+2

As the students become proficient it is important to expose them to a variety of rules including more complex linear relations e.g. $3n + 1$, $\frac{1}{2}n - 2$. Exposure to non-linear relations is important in helping students understand that a broader range of relations exists, e.g. simple exponents 2^n, and simple quadratics, e.g. n^2. It also requires students to make critical choices about whether they can find direct rules or they need to use recursive rules.

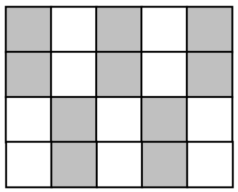
Opportunities also exist in the generalisations that arise from operations on numbers. These generalizations are based on classifications of the numbers. For example, consider the situation where a student is analyzing this calculator game:

Two players have a calculator each.
Without seeing the other person's calculator
they enter one of the digits, 1, 2, 3, 4, …, 9.
The digits are multiplied together mentally, e.g. 6 × 7 = 42*.*
If the product (answer) is even Player A wins.
If the product is odd Player B wins.
Would you rather be Player A or Player B? Why?

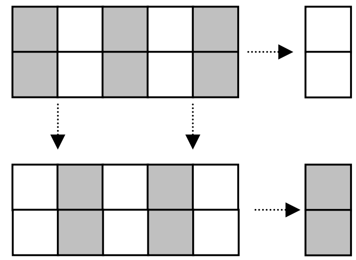
The students play the game sufficiently to determine that even product occur about twice as much as odd products.
Analysing why this occurs requires students to consider the products when odd and even factors multiply.
They may experiment with a few examples and come up with some generalizations:
3 × 5 = 15, 1 × 7 = 7, 9 × 3 = 27…odd × odd = odd
2 × 4 = 8, 6 × 6 = 36, 8 × 2 = 16…even × even = even
3 × 6 = 18, 4 × 7 = 28, 5 × 8 = 40…even × odd = even

Student might like to consider why these results are always true. This requires them to define an even number as one that is divisible by two with no remainder. In other words, a whole number of groups of two are made from any even counting number. Arrays give a good spatial model to show the multiplication relationships.

For example, consider even × odd = even:

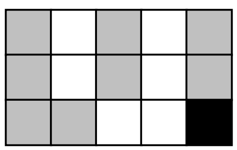


$4 \times 5 = 20$

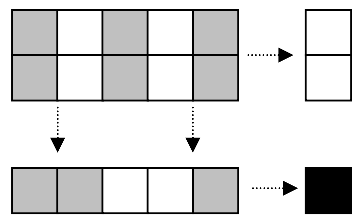


For any even × odd = even (as groups of two can always be formed by dividing along the direction of the even factor)

Consider odd × odd = odd:



$3 \times 5 = 15$



For any odd × odd = odd (as there is always a remaining one no matter which direction the groups of two are formed in)

True and false number sentences offer students the opportunity to conjecture, justify and generalise. These skills are important in thinking mathematically. Students are required to look at a statement of others and decide whether the statement is true of false. Initially these statements relate to specific numbers but may be broadened to include properties of number sets, e.g. odds and evens. Here are some examples:

Josh knows that 424 - 80 = 344. He thinks that 421 - 77 = 344 as well. Is Josh’s thinking true or false?

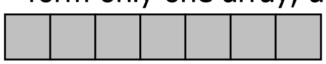
Casey knows that 400 + 300 = 700. She thinks that 700 – 296 = 404. Is Ani’s thinking true or false?

Mike says: “Multiply any odd number by any even number. The answer is always even. True or false?”

Ruka says: “The answer to 16 x 25 = □ is the same as 4 x 100. True or False?”

At Level Three students should begin to explore the properties of multiples and primes. For example, “Are the multiples of three even or odd?” From the dice game we should expect the multiples of three (odd number) to be both even and odd as the other factor varies from even to odd, e.g. 1 × 3 = 3 (odd), 2 × 3 = 6 (even), 3 × 3 = 9 (odd), 4 × 3 = 12 (even),… Students might look for similarities and differences in multiples. This leads to divisibility rules. For example, the multiples of five are 5, 10, 15, 20, 25,… The ones digit of these numbers is always either a 0 or 5. The multiples of four are 4, 8, 12, 16, 20,… These numbers are always even but the ones digits can be 0, 2, 4, 6, or 8.

Some numbers have only two factors (numbers that divide into them). For example, seven has two factors (1 × 7), so does thirteen (1 × 13), and eleven (1 × 11). These numbers are called prime numbers (primes for short). Geometrically they form only one array, a line of width one:


1 × 7 = 7


1 × 13 = 13

Useful resources
Figure It Out (Learning Media)
Algebra Level 2-3, pages 1-24
Algebra Level 3, pages 1-24
Number Sense and Algebraic Thinking, Levels 3-4
For full references and learning outcomes for the Figure It Out books go to nzmaths:
Planning sheets for level 2-3 and level 3-4 transition are on the nzmaths.co.nz website:
http://www.nzmaths.co.nz/node/1917
[Comprehensive Teacher notes are provided for each student book. These notes have been distributed to schools and can also be accessed through http://www.tki.org.nz/r/maths/curriculum/figure/index_e.php
Numeracy Project Book 9: Teaching Number through Measurement, Geometry, Algebra, and Statistics, pages 30-40.
nzmaths.co.nz units (This website is sponsored by the Ministry of Education)
http://www.nzmaths.co.nz/node/369 (Algebra: Matchstick Patterns)
http://www.nzmaths.co.nz/node/373 (Algebra: Hundreds of Patterns)
http://www.nzmaths.co.nz/node/380 (Algebra: Properties of Operations)
http://www.nzmaths.co.nz/node/381 (Algebra: Building Patterns Constantly)
http://www.nzmaths.co.nz/node/386 (Algebra: Cups and Cubes)
Digital Learning Objects (These are accessed through the Ministry of Education Digi-Store and are the result of a collaborative project run by The Learning Federation, Australia)
http://www.nzmaths.co.nz/learningobjects/313/3
http://www.tki.org.nz/r/digistore/protected/objects/?id=1990&vers=1.0
Other Website links:
http://illuminations.nctm.org/ActivityDetail.aspx?ID=33
http://nlvm.usu.edu/en/nav/category_g_2_t_2.html
http://www.bbc.co.uk/education/mathsfile/gameswheel.html