Mathematics in the New Zealand Curriculum
Second Tier

Strand: Number and Algebra   Thread: Patterns and Relationships   Level: Two

Achievement Objectives:

- Generalise that whole numbers can be partitioned in many ways.
- Find rules for the next member in a sequential pattern.

Exemplars of student performance:

Exemplar One:

Jamie has to write many names for the number sixteen. He organizes his expressions using patterns:

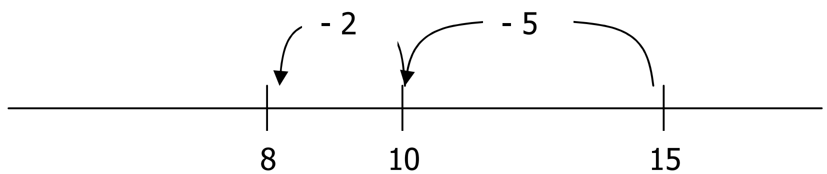
0 + 16	16 – 0	8 + 8
1 + 15	17 – 1	4 + 4 + 4 + 4
2 + 14	18 – 2	2 + 2 + 2 + 2 + 2 + 2 + 2 + 2
…	…	
14 + 2	30 -14	
15 + 1	31 - 15	
0 + 16	32 - 16	

Jamie is demonstrating Level Two achievement because he recognizes that the number sixteen can be partitioned in many ways and that there are connections between these partitions. For example, he knows that addends can be halved to create the 8 + 8 patterns and, for pairs of addends, adding one to an addend means ones must be taken off the other addend to keep the total invariant, e.. 2 + 14 = 3 + 13.

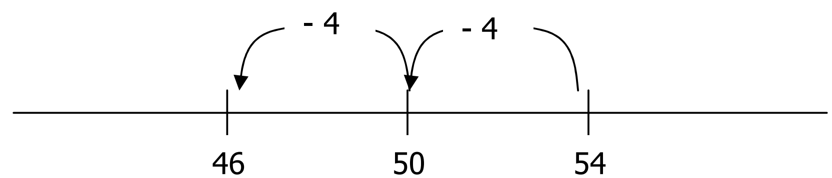
Exemplar Two:

Anakiwa is working out how Melinda solves simple subtraction problems. She has three examples of Melinda’s work to look at:

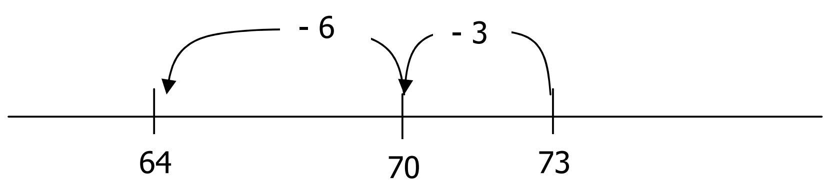
15 – 7 = □ as, 15 – 5 = 10, 10 – 2 = 8.



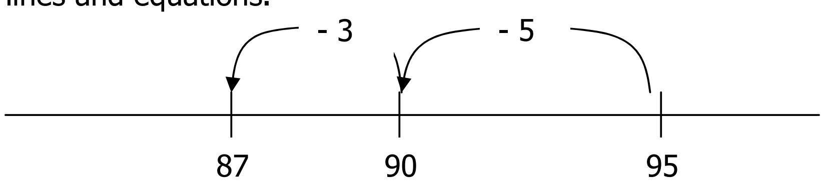
54 – 8 = □ as, 54 – 4= 50, 50 – 4 = 46.

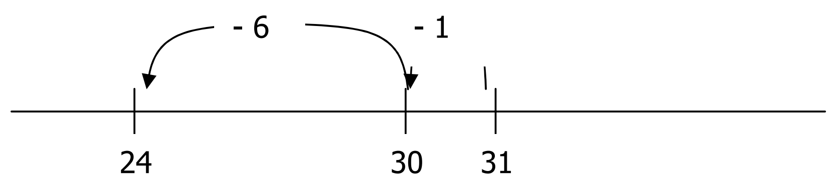


73 – 9 = □ as, 73 – 3= 70, 70 – 6 = 64.



Anikiwa is asked to show how Melinda would solve 95 – 8 = □ and 31 – 7 =□ . She shows her answers using empty number lines and equations.





95 – 8 = 95 – 5 – 3

= 87

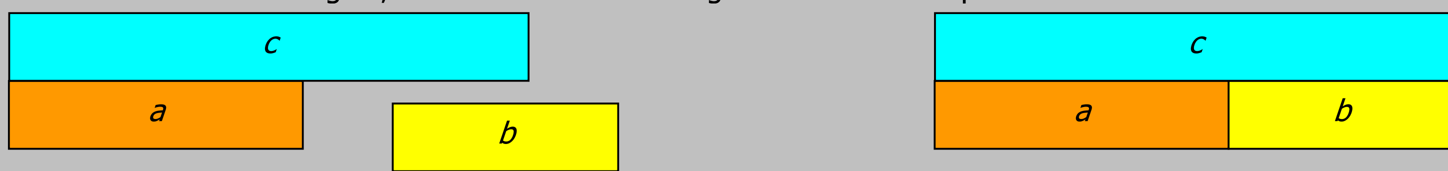
31 – 7 = 31 – 1 - 6

= 24

Anikiwa is demonstrating Level Two achievement because she uses her ability to split numbers to interpret another person's additive part-whole strategy. She knows partitions of numbers to ten, e.g. 10 – 3 = 7, 8 – 5 = 3, and simple place value partitioning, e.g. 54 – 4 = 50, and uses this knowledge to implement the "back through ten' strategy.

Exemplar Three:
Wesley solves this word problem:
There are 106 cattle in the yard. A truck takes 54 of them away. How many cattle are left.
He calculates 100 – 50 = 50 and 6 – 4 = 2 so the answer is 52. His teacher asks how he could check to see that his answer is correct. Wesley calculates 52 + 54 = 106 and confirms his answer is correct.

Wesley's answer shows achievement at Level Two because he is able to use addition, the inverse operation, to check his subtraction calculation. In doing so, he shows understanding of this relationship:



If $c - b = a$ then $a + b = c$

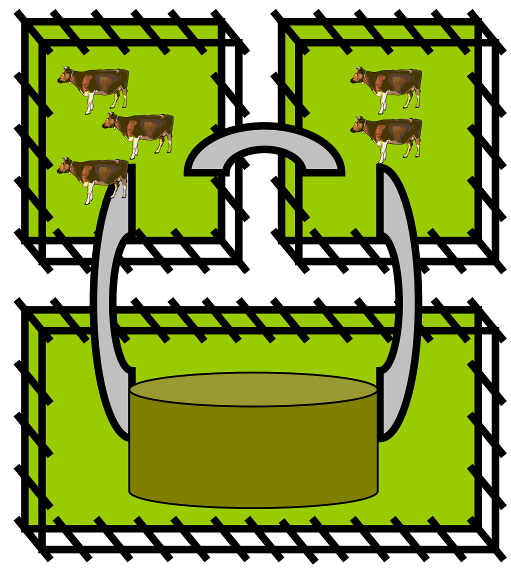
Note: The algebraic symbols used here are for teacher information and this not intended to be learned by students at level Two.

Exemplar Four:
Kayla has to find the unknowns in these equations:

7 + □ = 5 + 7	She knows that the box must be five since the order of the addends does not change the sum (commutative property)
8 + 6 = 10 + □	She notices that 10 is two more that eight so the box must be two less than six, four.
100 + □ = 18 + 99	She notices that 99 is one less than 100 so the box must be one less than 18, 17.
17 – 10 + 1 = 17 - □	She knows that the box must be 9 since 10 – 1 = 9.

Kayla's answers show Level Two achievement because she is able to operate relationally on the equations and does not need to calculate the total on one side of equals to find the missing number. In doing so she shows an understanding of the equals sign as meaning "the same as". Kayla shows understanding of the commutative property for addition, how the total remains unchanged as one addend increases while the other decreases by the same amount, and compensation can be used in simple subtraction problems.

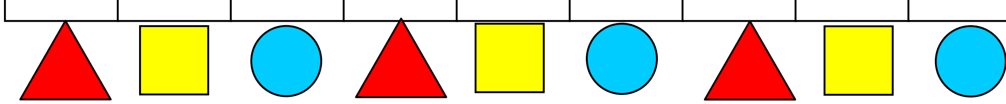
Exemplar Five:
Sarat is told that there are 12 cows on the farm in total.
She is asked, "How many cows are in the barn?"
Sarat reasons, "Three and two is five. Five cows are seen.
Five and five is ten so five and seven is twelve. There are seven cows in the barn."



Sarat's thinking shows achievement at Level Two because she recognizes that problem requires her to partition twelve into five and the unknown, $5 + \square = 12$.
She uses part-whole knowledge of 5, 10, and 12 to find her solution, $2 + 3 = 5$, $5 + 5 = 10$. $5 + 7 = 12$.

Exemplar Six:
Quentin identifies the repeating element in this pattern, "It goes red triangle, yellow square, blue circle."

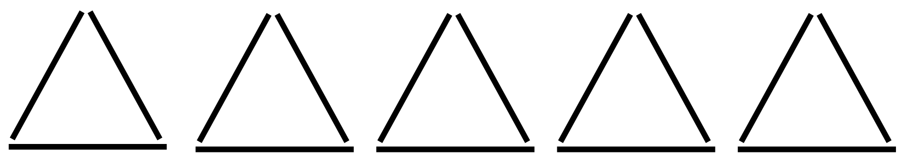
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15



Asked to predict what shape would match the number 24, he reasons, "A blue circle would be shape twelve. Twelve and twelve is twenty-four so under twenty-four would be a blue circle as well."

Quentin's thinking shows he has achieved Level Two because he can use additive partitioning, i.e. 9 + 3 = 12, 12 + 12 = 24, to predict members of a sequence. This shows that she is able to find a repeating element and use this to establish a relation between the objects and the matching skip counting sequence.

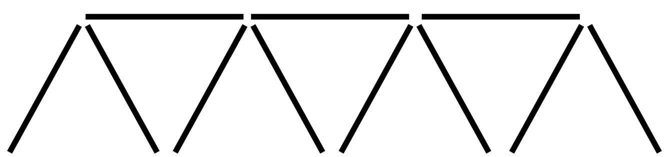
Exemplar Seven:
Delia and Don predict the number of matches needed to make ten triangles in this pattern.



Delia writes 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30. She works out the answer by repeatedly adding three.
Don counts the number of matches for five triangles using 6 + 6 + 3 = 15. He doubles 15 to get his answer.

Both Delia and Don's strategies show they have achieved Level Two as they are able to use repeated addition and doubling to predict further members of a sequential pattern. In doing so, they establish a repeating element that is counted.

Exemplar Eight:
Piri and Unca predict the number of matches needed to make ten triangles in this pattern.



Piri constructs a table, recognizes that two matches are added for each additional triangle and uses repeated addition to find a

solution.

Triangles	1	2	3	3	5	6	7	8	9	10
Matches	3	5	7	9	11	13	15	17	19	21

+2 +2 +2 +2

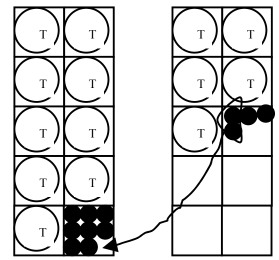
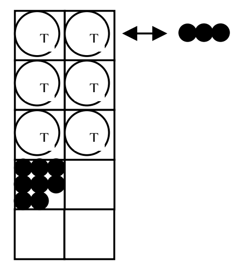
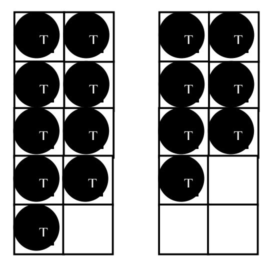
Unca writes: 9 + 2 = 11, 11 + 2 = 13, 13 + 2 = 15, …, 19 + 2 = 21.

Both Piri and Unca are able to develop an additive rule for continuing the pattern. The rules are recursive, that is, they tell what to do to the previous term to get the next. This means that the students have identified a repeating element, two matches, in the pattern. The creation of simple, additive recursive rules shows achievement at Level Two.

Important Teaching Ideas

The main focus of algebra at Level two is generalization of the properties for addition and subtraction and use of additive strategies to predict members of a relation. A relation is a set of ordered pairs, e.g. (1, 3), (2, 5), (3, 7)…, that describe a relationship between two variables, that is counts or measurements that change value. (1, 3), (2, 5), (3, 7)…could be used to describe the relationship between the number of triangles and the number of matches in the pattern shown in Exemplar Eight. The first number in any pair is the number of triangles, the second number gives the corresponding number of matches.

Generalisation of the properties for addition and subtraction involves students reflecting on the strategies they use to solve number problems. Strong links need to be made between the strategies used on a particular problem and the actual quantities involved. Representations, particularly physical materials, are vital aids in creating these connections. Interpretation of the strategies of other students is a sign that the properties have been generalised. A summary of key properties is given below:

Additive Property	Given Strategy	Representation	Missing unknown
Compensation for addition	T solves 98 + 54 = □ as 100 + 52 = 152. How would T solve 34 + 49?		What does ✡ represent in each equation? 9 + 7 = 10 + ✡ 67 + ✡ = 70 + 5
Commutative Addition	K solves 3 + 68 = □ as 68 + 3 = 71. How would K solve 5 + 476 = □?		What does ✡ represent in each equation? 3 + 69 = 69 + ✡ ✡ + 7 = 7 + 108
Addition and Subtraction as Inverses	M solves 90 + 80 = 170 so 170 - 80 = 90. How would M solve 33 – 18 = □ from 15 + 18 = 33?		What does ✡ represent in each equation? 16 + 7 = 23 so 23 – ✡ = 7 21 – 9 = 12 so ✡ + 12 = 21

Relations can be developed through equations, number patterns and spatial patterns. Consider these pattern of equations.

1 + 1 = 2	1 + 4 = 2 + 3	3 − 2 = 1
2 + 2 = 4	2 + 5 = 3 + 4	4 − 3 = 1
3 + 3 = 6	3 + 6 = 4 + 5	5 − 4 = 1

Students can extend these patterns at three levels of difficulty:

1. Write more equations in these patterns carrying on from the equations that are there.
2. Predict an equation in each pattern that is "a long way down".
3. Make up a rule for all equations in the pattern.

Writing more equations, as in number 1, requires recursive thinking. Students look down the equations to find the missing values. For example, to extend beyond 3 + 3 = 6 they add one onto each three and two onto six to get 4 + 4 = 8. Predicting an equation further down requires relational thinking across the equals sign. So the student might realize in the 3 + 3 = 6 that the right-hand sum is always twice the addends and that these addends are always the same. They choose a starting number, say 100, and apply the relationships to get 100 + 100 = 200. Extending this to finding a general rule requires students to accept the idea of a variable, that a value that can change relative to other variables. Encourage students to represent the rules diagrammatically, in words, and in invented equations. For example:

The first number is the same as the second number.
The answer is double the first number or double the second number.

<table>
<tr><td>First number</td><td>Second number</td></tr>
<tr><td colspan="2">Answer</td></tr>
</table>

☐ + ☐ = 2 × ☐ (☐ is the first and second number)

Students invent their own ways to record their generalizations. Link these recording methods and discuss the efficiency of recording This readily leads to formal algebraic notation though this is not required in any form at Level Two.

Develop students' competence with finding the rules for relations using contexts such as the number machine. Imagine numbers inputted into a machine, the machine does some operation on the input numbers and returns them as output numbers. It is best that the input numbers are not sequential as this encourages relational thinking between the input and output numbers. At Level Two students work mainly with addition and subtraction rules but simply multiply and divide rules as well as combinations of the four operations are important extensions in preparations for later levels.

Here are some examples of possible input/output rules. *The algebraic notation is for teacher information only*.

In	Out
5	7
2	4
10	12
8	10
n	n+2

In	Out
6	3
3	0
11	8
9	6
n	n-3

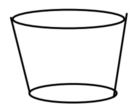
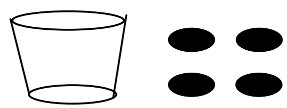
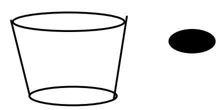
In	Out
2	4
6	12
5	10
12	24
n	2n

In	Out
4	20
7	35
11	55
2	10
n	5n

In	Out
6	3
14	7
2	1
5	2½
n	½n

In	Out
3	5
7	13
1	1
5	9
n	2n-1

Concepts of inverse operations, operations that "undo" one another, develop through activities like "Choose any number" puzzles. In the example below the puzzle steps are shown verbally, through a cups and counters model, and in algebraic notation. The cup represents any number chosen initially and the counters represent ones. The algebraic symbols are for teacher information only.

Words	Cups and counters model	Algebraic symbols
Choose any number		n
Add four		$n + 4$
Take away three		$n + 1$
Take away the number first thought of		1

Focus students on how the puzzle works by using operations that undo one another, i.e. add four take away three, choose a number then take it away.

True and false number sentences offer students the opportunity to conjecture, justify and generalise. These skills are important in thinking mathematically. Students are required to look at a statement of others and decide whether the statement is true of false. Initially these statements relate to specific numbers but may be broadened to include properties of number sets, e.g. odds and evens. Here are some examples:

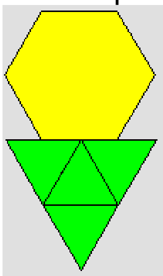
Liam knows that 48 + 56 = 104. He thinks that 46 + 54 = 100. Is Liam's thinking true or false?

Ani knows that 26 – 8 = 18. She thinks that 26 – 9 = 19. Is Ani's thinking true or false?

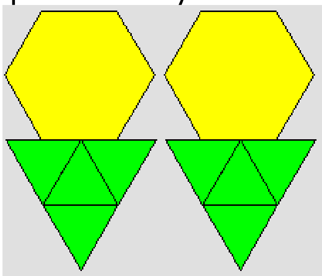
Marija says: "Subtract any odd number form any odd number. The answer is always even. True or false?"

Ruka says: "If a number has 0 or 5 as its ones digit then it is in the five times table. True or False?"

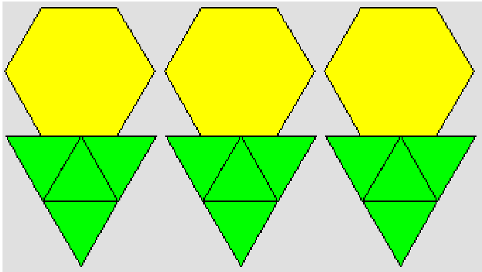
Sequential and repeating patterns provide opportunities for students to form relations between shapes and the set of ordinal numbers (see Exemplar Six) and between variables within a pattern (see Exemplars Seven and Eight). Identification of a repeating element in the pattern is important for systematic counting and leads to additive recursive rules. Consider this pattern:



1 Icecream



2 Icecreams



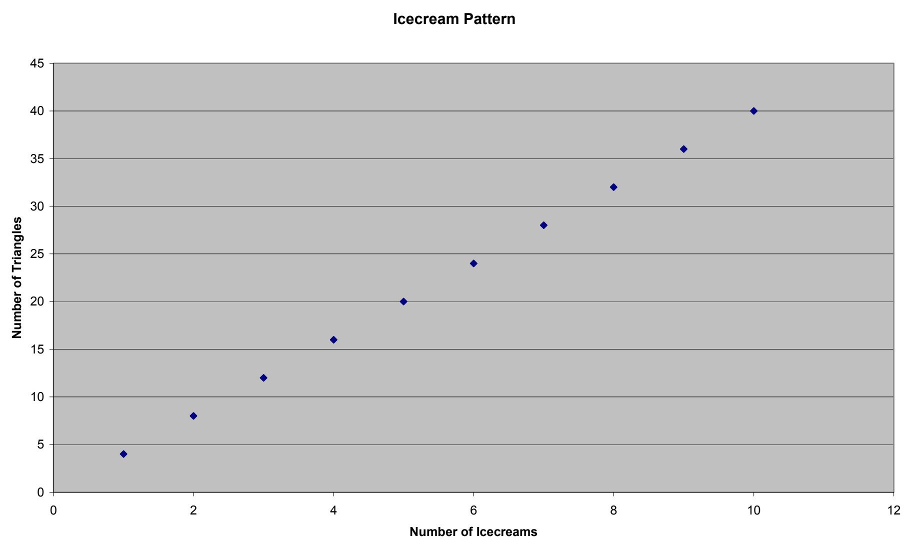
3 Icecreams

In answering the question, "How many triangles will be in 10 icecreams?" students identify a repeating element, four triangles added with every extra icecream. They may use this to find the solution through skip counting or repeated addition, i.e. 4, 8, 12, 16, …, 40, or 4 + 4 = 8, 8 + 4 = 12, 12 + 4 = 16, etc. These are recursive additive rules because they describe what to do to the

previous term to get the next. Given access to multiplication facts students may use a direct multiplicative rule, 10 × 4 = 40, ten lots of four triangles gives forty triangles.
It is important t expose students to multiple representations of relations as preparation for later levels. Tables, ordered pairs and graphs are standard representations:

Number of icecreams	1	2	3	4	5	6	7	8	9	10
Number of triangles	4	8	12	16	20	24	28	32	36	40

The set of ordered pairs is, {(1,4), (2,8), (3,12), (4, 16)…}



Note that the points on the graph lie on a straight line. This means the relation, triangles to icecreams, is linear, it goes up by a constant difference. That difference, four, is the slope of the line, four up for every one across.

Useful resources

Planning sheets for level 1-2 transition can be found on the nzmaths.co.nz.
http://www.nzmaths.co.nz/node/1917

Numeracy Project Book 9: Teaching Number through Measurement, Geometry, Algebra, and Statistics, pages 30-40.
nzmaths.co.nz units (This website is sponsored by the Ministry of Education)
http://www.nzmaths.co.nz/node/361 (Algebra: Staircases)
http://www.nzmaths.co.nz/node/362 (Algebra: Cuisenaire Mats)
http://www.nzmaths.co.nz/node/363 (Algebra: Letter Patterns)
http://www.nzmaths.co.nz/node/367 (Algebra: Supermarket Displays)

Digital Learning Objects (These are accessed through the Ministry of Education Digi-Store and are the result of a collaborative project run by The Learning Federation, Australia)
http://www.nzmaths.co.nz/learningobjects/316/2
http://www.nzmaths.co.nz/learningobjects/313/2

Other Website links:
http://illuminations.nctm.org/Activities.aspx?grade=all
http://nlvm.usu.edu/en/nav/category_g_1_t_2.html
http://www.bbc.co.uk/education/mathsfile/gameswheel.html