Supporting the Additive Thinking of Year 5–6 Students: The Equal Additions Strategy

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This study set out to investigate what actually happens in classrooms as teachers use the Numeracy Development Projects (NDP) resource materials to build students' repertoire of part–whole strategies for solving addition and subtraction problems. Nine teachers each worked with a group of students in their class who were judged to be ready for instruction in advanced additive thinking. They were observed teaching equal additions as a possible strategy for solving comparison problems in subtraction. On the whole, the teachers stuck pretty closely to the Equal Additions lesson described in NDP Book 5 (pp. 38–39). Despite their commitment to conceptual teaching, many of the teachers ended up teaching the equal-additions strategy in a procedural manner and did not encourage their students to weigh up the efficiency of this strategy when compared with other possible solution strategies. The availability of more background information about the key ideas underpinning the particular lesson and information about different problem structures for addition and subtraction problems would have benefitted the teachers.

Introduction

It is almost a decade since the national Literacy and Numeracy Strategy was launched (see Ministry of Education, 2001). The strategy was designed to raise students' achievement in mathematics (and language) and build teachers' professional capacity to teach mathematics (and language) (Ministry of Education, 2001). The Numeracy Development Projects (NDP) are a major government initiative designed to address both these components in relation to numeracy learning (Anthony & Hunter, 2005; Bobis, Clarke, Clarke, et al., 2005; Young-Loveridge, 2004, 2005, 2006, 2007, 2008, 2009). The Number Framework and associated diagnostic assessment tools were designed to help teachers make decisions about the nature of students' knowledge and understanding in relation to a sequence of learning progressions in number (Ministry of Education, 2008a, 2008b). A major goal for students is for them to understand the concept of additive composition, the way numbers can be split into parts and recombined (part–whole thinking) to solve problems more easily. Initially, students need to learn about this within the context of addition and subtraction processes. A distinction is made between those who are just learning to split and recombine single-digit quantities using a limited range of strategies (stage 5 early additive part–whole thinkers) and those who can do this with multi-digit quantities, making flexible use of a full range of strategies (stage 6 advanced additive part–whole thinkers).

The release of *The New Zealand Curriculum* (Ministry of Education, 2007) and *Mathematics Standards for Years 1–8* (Ministry of Education, 2009) has embedded the Number Framework within the system of expected outcomes that apply to all students in New Zealand schools. The goal is that by the end of year 4, students are able to split and recombine small numbers (stage 5), and by the end of year 6, students are able to choose from a range of strategies to solve multi-digit addition and subtraction problems (stage 6). The recent mathematics standards poster (Ministry of Education, 2010) indicates that students in the year prior to this should have an emerging capability (that is, early stage 5 by the end of year 3 and early stage 6 by the end of year 5). Hence it is vital that teachers have the necessary personal understanding of mathematics and of how to teach these mathematical ideas to students (pedagogical content knowledge [PCK]) so that they are able to support their students' learning in this way.

The NDP professional development programme for teachers working at the year 5–6 level was for just one year. That may have been enough for teachers to get their heads around the components of the NDP approach, such as the Number Framework, the assessment tools, the teaching model, and the published resources to support their teaching. However, it is questionable whether, within one year, teachers would also have been able to develop a deep and connected understanding of the complexity and subtleties of addition and subtraction as well as shift from a *calculational* to a *conceptual orientation* (Philipp, 2007). Virtually all primary schools have now had the opportunity to participate in NDP professional development programmes. The Ministry of Education's emphasis now is on in-depth sustainability of the NDP approach to teaching mathematics.

Data from year 6 students has shown that by the end of the year in which their teachers participated in NDP professional development, fewer than half of the students at this level were at stage 6 or higher (between 41% and 49% over the period 2005–2008; see Young-Loveridge, 2005, 2006, 2007, 2008, 2009). This is well short of the majority of students (75%–80%) expected to be at stage 6 or higher by the end of year 6 (Ministry of Education, n.d.). The proportion of year 5 students at stage 6 is considerably lower (between 22% and 34% over the period 2005–2008).

Several writers have emphasised the importance of students having an appreciation of the structure of mathematics (Bobis, Mulligan, & Lowrie, 2008; Lambdin & Walcott, 2007; Mulligan & Mitchelmore, 1997, 2009; Mulligan, Prescott, & Mitchelmore, 2004; Young-Loveridge & Mills, 2009a, 2009b). Particularly relevant to additive thinking is place value (the meaning of a digit within a multi-digit numeral by virtue of its position or place within the number). The base-ten structure of the number system dictates that 9 is the largest digit in any position within a multi-digit number. Once there are more than nine objects, the digit in the adjacent position (for example, tens) must "roll over" and the digit in the "ones" position begins incrementing again from zero. Students must learn about what combination of numbers makes ten (for example, 3 and 7) and then generalise this information to larger numbers to make other whole decades (for example, 53 and 7 makes 60, or 53 and 27 makes 80). Eventually, they also learn that when 1 is added to 99, another "rollover" occurs to create 100 and the digits in the "ones" and "tens" positions go back to zero.

This study set out to investigate what actually happens in classrooms as teachers use the NDP resource materials to build students' repertoire of part–whole strategies for solving addition and subtraction problems.

Method

Participants

Nine teachers (seven female and two male) of year 5–6 students from four schools ranging in decile rank from 2 to 10 participated in the study. One teacher from each school had previously worked with the researchers, and that teacher agreed to ask the other teacher/s working at the same level if they would agree to be involved in the study. Teachers' classroom experience ranged from two to approximately 25 years. Experience in working with the NDP approach ranged from two to about seven years. The focus of this paper is mostly on teachers A, B, C, and D (with 6–25 years of classroom teaching and 5–6 years of teaching with the NDP approach), whose transcribed lessons were available at the time of writing.

Procedure

Students were given a paper-and-pencil assessment prior to their teachers teaching the first lesson (Savings Hundreds; see Ministry of Education, 2008c, p. 32), then a similar assessment after the

third lesson (A Balancing Act; Ministry of Education, 2008c, p. 40). The lesson that is the focus of this study is Equal Additions (see Ministry of Education, 2008c, pp. 38–39), the second of the three lessons observed. During the lesson, the teacher wore a portable digital audio-recorder attached to a flexible belt, with a lapel microphone to pick up their language to the students (and some responses from students who were close to the teacher). Students wore name tags so that the researchers could identify particular students in their observational records. The researchers observed the lesson and noted non-verbal (contextual) information that could assist with the interpretation of the transcripts of audio-recordings. Actions with materials, written recording in the group recording book, and students' individual mathematics books were photographed to capture some of this non-verbal information. Although the initial lesson, Saving Hundreds, is not the focus of this paper, it should be noted that the experience of working with missing addend problems, such as \$287 + ☐ = \$400, helped to familiarise the students and their teacher with what it was like to be observed and recorded by the researchers. Saving Hundreds also introduced the use of paper money, materials that were essential for the Equal Additions lesson. Another important aspect of the Saving Hundreds lesson was the exchanging of ten \$1 notes for one \$10 note, and vice versa, and this provided an important introduction to the contexts used for the Equal Additions lesson.

In the Equal Additions lesson, the first scenario is as follows:

> Problem: "Debbie has \$445 in her bank account, and her younger sister Christine has \$398. How much more money does Debbie have?"
>
> Make piles of \$445 and \$398. "Now suppose that Grandma gives Christine \$2 to give her a 'tidy' amount of money. To be fair, Grandma gives Debbie \$2 also." Discuss why 445 – 398 has the same answer as 447 – 400 and then record 445 – 398 = 47 on the board or modelling book.

The book then provides other examples of equations that can be turned into word problems and solved using materials (for example, paper money). It then gives further examples of equations that can be turned into word problems and solved using number properties.

Results

The transcripts for the Equal Additions lesson were analysed according to the content of each teacher's lesson. Most of the teachers drew students' attention to the learning intention for the lesson (see Figure 1). Teacher C appeared to teach this particular lesson very effectively. An examination of the transcript revealed several key features of teacher C's lesson that were not found to the same extent in the other teachers' lessons. These features are described in following sections.

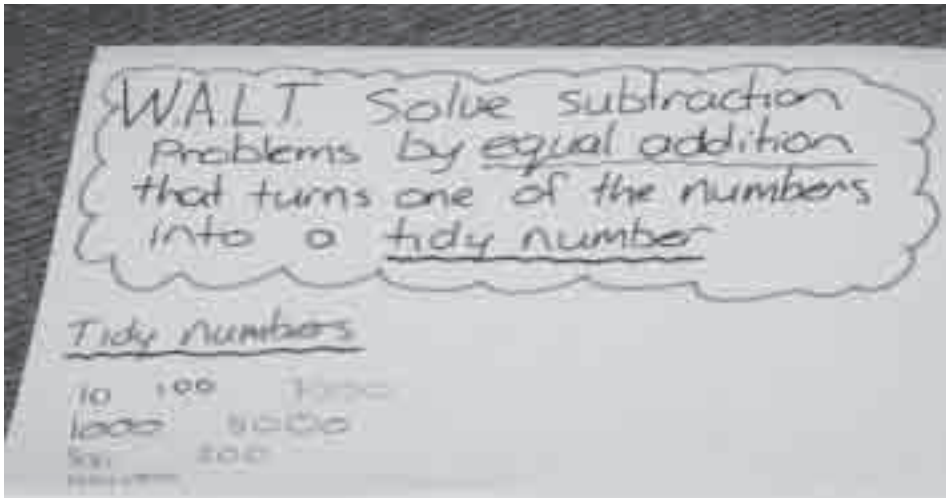

Figure 1. Teacher B's group recording book showing the learning intention (W.A.L.T.: We Are Learning To)

Personalising the Problem for the Students

Some teachers (for example, teacher C) changed the names of the students in the problem to those of students in the group they were working with and got the group to act out the scenario with a different person responsible for each denomination of the money. With subsequent problems, the names of other students in the group were used. Most of the teachers rotated responsibility for managing each denomination of the money so that all students became involved in the problems (see Figure 2).

Fairness

Teacher C emphasised the idea of fairness as a justification for giving the same amount of money to the two students who represented the subtrahend (the number being subtracted) and the minuend (the number from which the subtrahend is subtracted; note that the teacher did not actually use the terms minuend and subtrahend). Teacher B also commented on the importance of fairness as a reason for using equal additions to solve the problem. His scenario had Grandma giving $2 to each child as a gift that had to be kept fair. Teacher C asked the students to suggest a reason why Grandma might give $2 to one child and accepted the suggestion that it was for mowing the lawn, so to give $2 to the other child (who had apparently done no work) did not seem quite fair. This highlighted the importance of making sure that contexts are plausible.

Figure 2. Teacher A's group recording book and money used in the lesson

Proving that the Difference Remained Constant

At one point, teacher C checked to see if the students believed that the difference between the minuend and the subtrahend after the equal additions was the same as it had been before:

> Would that difference between these two numbers [445 and 398] have been 47?

Some students disagreed, expressing doubt about the magnitude of the difference being unaffected by the equal additions process. Teacher C then asked:

> So what's the difference between those two numbers? What do you think, S?

S (C4) responded "49". Teacher C then asked another student, C (C3), who responded "47".

Teacher C then got the students to check the difference by building the subtrahend up until it equalled the minuend and to keep track of how much money they added. The student holding the initial $398 was given $2 by the bank, and that amount ($2) was then recorded. The resulting ten $1 notes were taken away and traded for one $10 that was added to the pile of nine $10 notes. The resulting ten $10 notes were then taken away and traded for one $100 note, which was then added to the pile of

three $100 notes. The students were then asked how much money they would need to get to $445. All appeared to agree that another $45 was needed to build up from $400 to $445. Teacher C then commented:

> Let me get this straight. The difference between this amount here [398] and this amount here [445] was $47. Nana gave you both $2 more, and the difference between those two amounts [400 and 447] is …

One student responded, "It would be exactly the same."

Later, one student raised the possibility of turning the minuend into a "tidy number" rather than the subtrahend. Teacher C supported the student in investigating whether it mattered which number was turned into a tidy number. Eventually, the student came to recognise that it was only when the subtrahend (the amount being taken away) was made into a tidy number that the problem became easy to work out (see Figure 3).

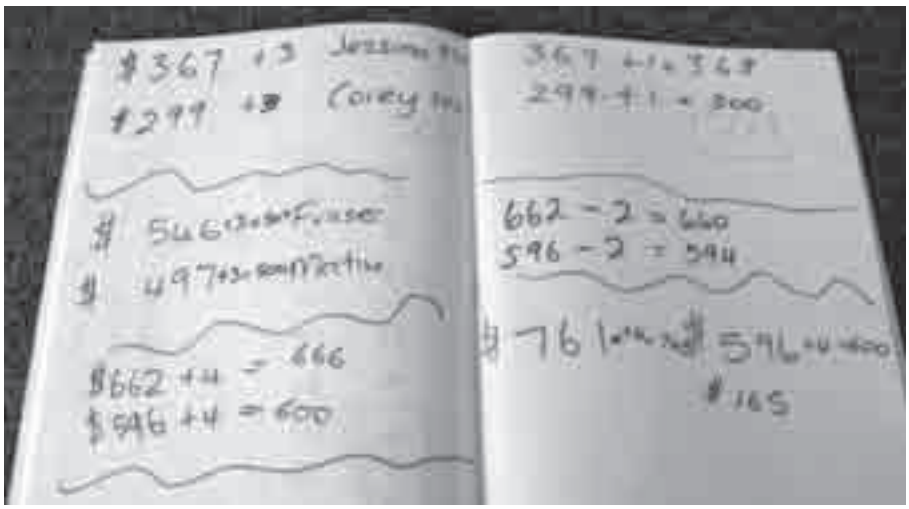

Figure 3. Teacher C's group recording book with students' recording of equal additions

Not all the teachers realised that making the subtrahend tidy was the most efficient way to apply the equal additions strategy. It became clear during the lesson that teacher D did not understand why making the minuend a tidy number would be less helpful than making the subtrahend a tidy number. This was evident in her statement that:

> They're both right, so it doesn't matter which number we turn into a tidy number.

Although teacher D was technically correct, adding 5 to both numbers (which rounded 445 up to 450 and 398 to 403) meant that the subtraction of 403 from 450 still involved the challenge of regrouping and so was not as easy to work out as 400 from 447. Hence, making the subtrahend tidy makes the subtraction problem much easier to work out.

Consistency of Language and Mathematical Structure

Although it was not clear whether teacher C understood about different problem structures, she was very careful in her use of mathematical language and special terminology with the students.

Teacher D took the first example from the book (see description above), which was structured as a *compare* (difference) problem and turned it into a *separate* problem. She said:

> Right, I had $445 right, ... K asked me if she could have a loan of $398 and being the giving, caring person that I am, I said sure. How much money did I have left over? Right, I want you to think about the tidy numbers, using tidy numbers.

One student (S: D1) was concerned that if 2 was added to one number, it needed to be taken off later. Teacher D explained to S (D1):

> S, what I think you've been confused with is if we did it to one of these numbers, if we added 2 to one number, then yes, we do have to take it away – but we did it to both numbers. If we just added 2 to 398 and 445 the same, then yes, we would have to take that 2 away, but because we do the same treatment to both numbers, the gap remains the same.

Several students commented that they had got lost, so teacher D then decided to bring the lesson to a close because they had run out of time for further explanations. (Note: In the post-lesson interview, when teacher D was asked what she would do to follow up on the lesson at a later date, she did not mention an intention to clarify the confusions that emerged during the Equal Additions lesson.)

Teacher B introduced his lesson by sharing with the group how, in preparing for this lesson, his own mathematics had been extended.

> This is one of these really cool exercises. Now I mentioned before that, since doing this, my understanding of maths has really improved. What this next lesson is [is an] actually really cool lesson for an area that I think we've got a bit of a weakness in as a class, looking at one particular type of operation. Now, so what we are going to do is, we're going to look at ... how to solve subtraction problems by equal addition that turns one of the numbers into a tidy number.

Teacher B then asked the students, "What sort of problem are we looking at?"

W (B1) suggested a missing addend structure: 398 plus what equals 445. Teacher B would not accept the missing addend structure because the learning intention focused on subtraction. He said:

> Oh okay, so, W, you've gone for that first one, reversing strategy, so you've gone for 398 plus what equals 445, yeah. If we just look at the learning intention, which is to solve subtraction problems by using equal addition, are we using a subtraction problem here?

One student answered "No". Teacher B continued:

> Is this still a good strategy? Yep, but we're going to look at just using subtraction, so what problem am I going to write down here to show subtraction?

Another student suggested 445 take away 398. Teacher B affirmed that response:

> Nice, so we are going to use 445 take away 398 equals ... We think it might be 47? [suggested earlier by one of the other students]

Teacher B then asked whether adding 2 to both numbers would change the answer. Some students thought it would increase the answer by 4, but others believed that the answer was still the same. Teacher B tried to get those students to explain why:

> What are we actually looking at, we're looking at what? ... So when I give the answer, what's the answer? Okay, if we take the answer, we say, let's say that's 47, we're happy that it's 47. What does the 47 actually mean?

One student suggested "Numbers?", to which teacher B responded, "Excellent, nice, okay."

There is further discussion, but it does not appear to produce what teacher B is wanting, so he explained:

> Okay, with subtraction, we're really looking at the difference between these two numbers, so the difference between 445 and 398 is 47, so we're just looking at difference. So the numbers here, you can change the numbers either way and it's not going to affect the outcome. Does that make sense?

At least one student agreed, but another was concerned about what happens if different amounts are added to different numbers. Teacher B responded:

> Ah now, good question. Will that affect the answer if you're not adding the same amount to each side, because you're looking at difference, but that's a good question, that's a very good question.

He then gave them another problem, but he did not stick consistently to either a *compare* or a *separate* structure.

> Okay, let's have a look. I'm going to give you another problem. Here you go. This time, W's got 367 apples and D would like to have some. He thought he could probably eat 299 apples 'cause he's sort of feeling a little bit hungry, he hasn't eaten for a while. So W started off with 367, and he is going to give D 299 of those 'cause he's quite generous. Now can you predict, now thinking about using that equal addition, will that help us solve the problem?

At least one student responded "Yes". Teacher B responded:

> Keeping in mind that we're looking at the difference between these two numbers, not necessarily the numbers themselves.

One student suggested that the answer was 68. When asked by teacher B how he did it, he responded:

> I gave each of them one more.

Teacher B then pressed for understanding:

> So just while we are doing this, but with F adding on one more, have we changed the difference between the numbers?

The students responded with both "Yes" and "No". Teacher B asked:

> We've changed the numbers, but have we changed the difference between the two numbers?

This time, the students knew they were expected to answer "No".

The Importance of Recording the Process Clearly

Teachers differed in the clarity of their recording. Teacher A recorded the equal-additions process vertically so that the adjusted equation was shown horizontally below the original equation (see Figure 4).

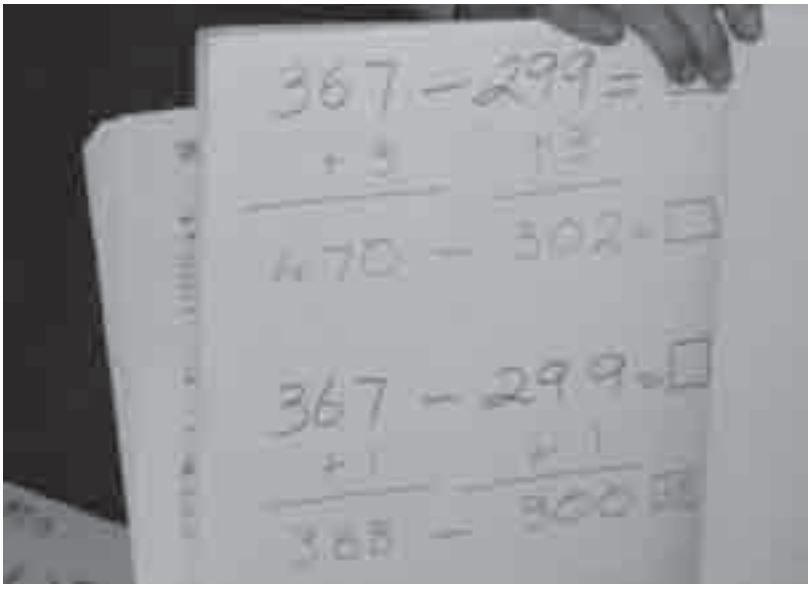

Figure 4. Photograph showing the "tidying" of the minuend (above) and the subtrahend (below) by Teacher A

In the Equal Additions lesson, the adjustments were so small (less than nine) that place value was not usually an issue. However, in the Saving Hundreds lesson, the researchers noticed how important it was to record each rounding-up step separately; for example, rounding up the "ones" digit to the nearest ten (7 + **3** = 10), then rounding the "tens" digit up to the nearest hundred (290 + **10** = 300), then working out how many more hundreds were needed (300 + **100** = 400). Finally, the three amounts needed to be added up to work out the total difference (**3** + **10** + **100** = **113**). It was important for following the step-by-step process that these three amounts were recorded separately. The researchers noticed that some teachers simply inserted the subsequent digits to the left of the one before, so that 3 became 13, which then became 113. This meant that it was not possible then to go back later and work out how the difference of 113 had been found; that is, it was the result of rounding 7 up (by 3) to 10, then rounding 290 up (by 10) to 300, and finally rounding 300 up (by 100) to 400.

Related to the need for care in recording was also the care needed in working with the materials. The researchers noticed one teacher who did not adhere to the rule that there could never be more than nine of any single denomination of the money. That teacher arranged for the person with the eight $1 notes to be given two more $1 notes to make ten altogether but did not ask the person who then had ten $1 notes to exchange them at the bank for one $10 note. The result was that the person with the ten $1 notes was asked to give them ("the ten") to the person holding the eight $10 notes (the "tens person"). The person with $10 notes then had a mixture of denominations, potentially a very confusing situation for students trying to understand place value.

Another important place value issue was the need to ensure that when students described the addition or subtraction process, they referred to the total value of the digits that they were operating with rather than just their "face" value. For example, with 900 – 298 (adjusted to 902 – 300), they needed to describe the subtraction of "three hundred from nine hundred" rather than "three from nine".

Discussion

The teaching of addition and subtraction was far more problematic than the researchers had anticipated. This seemed to be partly the result of teachers having a relatively superficial understanding of addition and subtraction processes and partly because Book 5 (Ministry of Education, 2008c) did not contain enough information to help teachers understand the underlying rationale behind each lesson. There was nothing in the description of the lesson to indicate that the problem structure for the missing-addend problems presented in the Equal Additions lesson was based on subtraction as difference (that is, the *compare* structure) rather than as "take away" (that is, the *separate* structure) (see Carpenter et al., 1999). None of the teachers seemed to be aware of the different problem structures that were possible for addition and subtraction, despite the fact that there has been considerable published work on classifying addition and subtraction problems according to "the types of action or relationships described in the problems" (Carpenter et al., 1999, p. 7). The classification scheme provides a structure for teachers to select problems that they could use for instruction and assists in the interpretation of students' solution strategies. Four classes of problems have been identified, including *join*, *separate*, *part–part–whole*, and *compare*. Join (addition) and separate (subtraction) both involve action, such as putting the objects in two collections together (join) or removing some objects from a collection (separate). Part–part–whole involves the static relationship between a collection and the parts that make up the collection. Compare involves comparisons between two different collections.

Another aspect of the classification focuses on which quantity is the unknown. For join or separate problems, the unknown quantity could be the start, the change, or the result. Traditionally, teachers have mainly used result-unknown problems, such as 8 + 5 = ☐. The missing-addend problem above ($287 + ☐ = $400) is an example of a join-change-unknown problem (that is, an addition

problem where the second addend is unknown). The NDP resources for addition and subtraction provide problems where the unknown is in different positions, but there is nothing to help teachers understand the classification system or why it is important for students to be given problems from each of the different categories (Ministry of Education, 2008c). For part–part–whole problems, the unknown quantity could be either a part or the whole. For compare problems, the unknown quantity could be the difference, one of the compared collections, or the so-called "referent" collection (Carpenter et al., 1999). The equal-additions strategy provides students with a very elegant solution strategy that is suitable for solving difference-unknown problems from among the various compare structures. It is also valuable for solving separate problems of the result-unknown type (for example, 9 – 6 = ☐) and any of the other problem types in which a part is unknown (for example, join problems of the change-unknown type).

It was interesting to note that on the post-lesson assessment, very few students chose to use the equal-additions strategy to solve the compare problem or one of the two subtraction equations, despite the subtrahend being close to a whole decade or century. This may have been because their teachers did not appreciate that the main point of the lesson was to understand that subtraction could be about difference (rather than "take away") and instead focused on making tidy numbers. Hence, neither teachers nor students valued the equal-additions strategy as a very simple and efficient way to solve subtraction problems with certain numbers. Unfortunately, the students had simply learned a rule for adding the same amount to both sides because they had been told to do so in order to round one quantity up to a tidy number and make the problem easier to solve. This is quite contrary to the goals of the NDP, which are to teach mathematics conceptually rather than procedurally. Most students continued to use a "bridging through ten/hundred" or the "reversibility" strategy to solve the problems in the post-lesson assessment. The students tended to have one favourite strategy that they tried to use for all problems, regardless of its suitability. This raises questions about whether students are actually being empowered to draw on a flexible range of strategies, or whether, instead, the formal written algorithm is simply being replaced by other algorithms that are applied equally mindlessly (Kamii, 1994; Pesek & Kirshner, 2000; Skemp, 2006). Teacher B commented on this in his interview:

> Interesting, G, even when you've been through and all the demonstration, still chose to use his own strategy, so yeah, it was interesting that he reverted back to something he felt comfortable with, so whether he didn't understand or didn't feel comfortable with what we were doing, I'm not too sure.

It would have been good to see more discussion among students about their solution strategies (Whitenack & Yackel, 2002). Hunter (2005, 2006) argues that students need to be encouraged to question each other to check understanding and to convince each other. Teachers tended to control their lessons quite tightly, but this may have been partly because they had been asked to teach a particular lesson as part of the research and they wanted to make sure that the lesson went the way they had anticipated it would. A similar comment could be made about the recording process: students were given few opportunities to record their solution strategies in the group recording book because their teachers tended to do it for them.

Teachers' and students' understanding of the role of the equals sign was an issue, and this was highlighted when the teachers taught a lesson (A Balancing Act) designed to help them appreciate the idea that the expression on the left-hand side of the equals sign should be the same as (or equivalent to) the expression on the right-hand side. For example, the learning intention for A Balancing Act is stated as: "I am learning that the answer on the left of the equals sign is the same as the answer on the right of the equals sign" (Ministry of Education, 2008c, p. 40). Just prior to the learning intention, the book states that "the equals sign means that the total on the left of sign is the same the total on

the right." The researchers observed that this learning intention seemed to lead directly to teachers asking the students to calculate the answer for the expression on each side of the equals sign. They did not ask the students to look at the expressions on the left and right of the equals sign and to decide on the missing value by looking at the change in values from left to right and the adjustment to the missing value needed to compensate for that change. This became particularly problematic when the numbers became larger. For example, in the problem 442 – 38 = ☐ – 39, it is not necessary to calculate that 442 – 38 is 404 in order to work out that if 38 is subtracted on the left but one more than 38 (that is, 39) is subtracted on the right, then the missing value must be one more than 442 to ensure that the difference between the two quantities on the left-hand side is the same as the difference between the two quantities on the right. Most of the teachers asked the students to calculate the answer on each side of the equation. Students need to be alerted to look at what adjustments need to be made to the missing value to compensate for changes in the values of quantities on the left and right of the equals sign. The researchers found that very few of the teachers suggested drawing arrows from the left of the equation to the right and recording the size of the change between a quantity on the left and a closely related quantity on the right.

A common problem for many of the teachers was to allow the steps in the process of recording addition or subtraction to "run on" so that the equation was no longer true and therefore did not balance. Teachers (and students) tended to see the equals sign as signalling that the result was following rather than as a statement about the equivalence of the sides of the equation. This became particularly obvious in the subsequent lesson on A Balancing Act. For example, for a problem asking "If John has $360 and Troy has $298, how much more money does John have than Troy?", it was common for students to first record "298 + 2 = 300". Then, instead of starting a new line and adding 60 onto 300 to get 360 (300 + 60 = 360), they recorded the second step of the process straight after the first step, thus violating the requirement of equality within the equation because the first part of statement was no longer true. That is, they recorded the process as 298 + 2 = 300 + 60 = 360. In doing so, they failed to recognise that 298 + 2 does ***not*** equal 360.

Teachers tended to stick very closely to the lesson description in the book, including beginning with the difference between two 3-digit numbers. None of them began with single-digit quantities to get across the initial idea of difference or the comparison between two quantities. None used materials like multilink blocks to model two towers of different heights (for example, 5 compared with 7) to show how the difference in height between the two towers would be unaffected by adding two blocks to each tower (7 compared with 9). None used a number line to show how the distance between two numbers on a number line is constant as long as the same quantity is added to both numbers. Even though Book 5 (Ministry of Education, 2008c) is a reasonably substantial book, there are relatively few lessons that deepen students' knowledge of partitioning or part–whole strategies to give them a full range of possible strategies for solving addition and subtraction problems (that is, to support them in becoming advanced additive part–whole thinkers). This is a particular problem for teachers who treat the NDP resource books as prescriptive (rather than illustrative).

It was interesting to note that none of the teachers mentioned the possibility of solving problems by equal subtraction rather than by equal addition if the minuend could more easily be turned into a tidy number by subtracting one or two from both numbers. It would have been helpful if Book 5 had given some examples of problems that could be solved efficiently by equal subtraction (for example, 489 – 206 = (489 – 6) – (206 – 6) = 483 – 200 = 283). That would encourage teachers to see the difference between the two numbers as being emphasised, as opposed to subtracting digit by digit across the three place value positions.

To support numeracy teaching, the NDP have provided the Number Framework, which shows the learning progressions within particular domains, along with diagnostic assessment tools and

a collection of resource materials. It may appear that teachers have everything they need in terms of resources to teach numeracy. However, what is missing from the NDP resource materials is the theoretical underpinning and variety of possible structures for addition and subtraction problems. What is also missing for a lot of teachers is a deep and connected understanding of the mathematics required for teaching with a conceptual emphasis; in other words, they lack pedagogical content knowledge (PCK) in mathematics (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Goya, 2006; Moch, 2004).

It is generally assumed by teachers that helping students learn to add and subtract is a very straightforward process. However, this overlooks the enormous complexity of these operations. We believe that there is a lack of appreciation of the challenge for teachers of coming to understand at a deep conceptual level what addition and subtraction with whole numbers is all about. As Bahr and de Garcia (2010) point out in the title of their book, "Elementary mathematics is anything but elementary."

Conclusions

A number of key recommendations have emerged from this research:

- Teachers need to build a deeper, more connected understanding of addition and subtraction processes as part of developing greater pedagogical content knowledge in mathematics.
- Teachers would benefit from being given additional support material outlining the variety of problem structures for addition and subtraction to which their students should be exposed.
- The purpose of the illustrative lessons in the NDP "pink" books need to be clearly articulated so that teachers understand the reason for including that particular lesson in their classroom mathematics programme.
- The learning intentions for some lessons need to be reworded to indicate their purpose more accurately. For example, the learning intention for the lesson entitled A Balancing Act needs to be reworded to avoid implying that students need to find the "answer" in order to solve the problem. Students need to be alerted to look at what adjustments should be made to the missing value to compensate for changes in the values of quantities on the left and right of the equals sign.
- It would be helpful for many teachers to have additional notes about the key ideas they need to understand before they teach the lessons (for example, the idea that subtraction can be about comparing quantities to find the difference between them).
- An understanding is needed that the *compare* structure for subtraction is about *difference* rather than about "take away" or separating a collection into parts.
- It is important that teachers are careful in the way they demonstrate their recording of equations, ensuring that expressions on either side of an equals sign are in fact equal. Hence the steps involved in solving a problem need to be recorded separately, on a new line for each step.
- Students need to be encouraged to discuss the usefulness of particular strategies to solve particular problems (or problems with particular numbers). The goal should be about making connections between different strategies and about considering the efficiency of particular strategies.
- Students should feel comfortable about questioning their peers and about justifying to them the particular strategies they have chosen to use. This is an important aspect of the mathematical communication that is now emphasised as part of mathematics education reform.

- Giving students responsibility for recording in a group recording book (rather than the teacher doing all the recording) is important to help engage the students in the problem-solving process.
- The so-called "modelling book" needs to be renamed the "group recording book" to avoid implying that it is a place where the teacher provides good examples or "models" to students. Students should be encouraged to take ownership of the group recording book to record their strategies and the discussions they have about the efficiency of those various strategies.
- The emphasis needs to be on building conceptual understanding rather than being trained in particular procedures. The dangers of simply replacing one algorithm with another needs to be made clear to teachers.
- The equal-additions strategy needs to be introduced to students using small (single-digit) quantities before moving on to larger numbers. The process needs to be modelled initially with small quantities, using linkable blocks to demonstrate that adding the same number to two quantities does not change the difference between them.
- The number line could also be used with students to show the constancy of the difference between two numbers when the same quantity is added to each number. A cardboard strip with clip-on pegs can be used to record the initial difference between the two numbers and then used again to check that the difference between those numbers has not changed after the same quantity is added to each number.
- Students need to be encouraged to explore the issue of whether tidying the subtrahend or the minuend is the most efficient strategy when using equal additions.
- When teachers use paper money to model addition and subtraction problems, they need to make a special point of emphasising that there can never be more than nine notes of any one denomination and that every ten notes of a particular denomination need to be exchanged for one note of the next denomination (for example, ten $1 notes for one $10, ten $10 notes for one $100 note).
- Students need to be encouraged, when describing operations, to refer to a digit in terms of its total value rather than simply its face value (for example, in 398, the 9 should be referred to as 90 and the 3 as 300).

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References

Anthony, G., & Hunter, R. (2005). A window into mathematics classrooms: Traditional to reform. *New Zealand Journal of Educational Studies, 40*, 25–43.

Bahr, D. L., & de Garcia, L. A. (2010). *Elementary mathematics is anything but elementary: Content and methods from a developmental perspective.* Belmont, CA: Wadsworth Cengage Learning.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator,* Fall, 14–22, 43–46.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, R., Young-Loveridge, J., & Gould, P. (2005). Supporting teachers in the development of young students' mathematical thinking: Three large scale cases. *Mathematics Education Research Journal, 16*(3), 27–57.

Bobis, J., Mulligan, J., & Lowrie, T. (2008). *Mathematics for students: Challenging students to think mathematically* (3rd ed.). Frenchs Forest, NSW: Pearson.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Students' mathematics: Cognitively guided instruction.* Portsmouth, NH: Heinemann.

Goya, S. (2006). Math education: Teaching for understanding: The critical need for skilled math teachers. *Phi Delta Kappan, 87*(5), 370–372.

Hunter, R. (2005). Reforming communication in the classroom: One teacher's journey of change. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds). *Building connections: Research, theory and practice.* (Proceedings of the28th annual conference of the Mathematics Education Research Group of Australasia, July, Melbourne, pp. 451–458.)

Hunter, R. (2006). Structuring the talk towards mathematical inquiry. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds), *Identities, cultures, and learning spaces.* (Proceedings of the 29th annual conference of the Mathematics education Research Group of Australasia, 1–5 July, Canberra). Adelaide, SA: MERGA.

Kamii, C., with Livingston, S. .J. (1994). *Young students continue to reinvent arithmetic – 3rd grade – Implications of Piaget's theory.* (Chapter 3: The harmful effects of algorithms) New York: Teachers College Press.

Lambdin, D. V., & Walcott, C. (2007). Changes through the years: Connections between psychological learning theories and the school mathematics curriculum. In W. G. Martin, M. E. Struchens, & P. C. Elliott (Eds), *The learning of mathematics: Sixty-ninth yearbook* (pp. 3–25). Reston, VA: National Council of Teachers of Mathematics.

Ministry of Education (2001). *Curriculum Update 45: The numeracy story.* Wellington: Ministry of Education.

Ministry of Education (2007). *The New Zealand Curriculum.* Wellington: Ministry of Education.

Ministry of Education (2008a). *Book 1: The Number Framework: Revised edition 2007.* Wellington: Ministry of Education.

Ministry of Education (2008b). *Book 2: The diagnostic interview.* Wellington: Ministry of Education.

Ministry of Education (2008c). *Book 5: Teaching addition, subtraction, and place value.* Wellington: Ministry of Education.

Ministry of Education (2009). *Mathematics Standards for Years 1–8.* Wellington: Ministry of Education.

Ministry of Education (2010). *Mathematics Standards for Years 1–8* (poster: item number 11475). Wellington: Ministry of Education.

Ministry of Education (n.d.). Curriculum expectations. Retrieved on 1 March 2010 from http://new.nzmaths.co.nz/using-expectations.

Moch, P. L. (2004). Demonstrating knowledge of mathematics: Necessary but *not* sufficient. *Curriculum and Teaching Dialogue, 6*(2), 125–130.

Mulligan, J., & Mitchelmore, M. (1997). Young students' intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28*(3), 309–330.

Mulligan, J., & Mitchelmore, M. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal, 21*(2), 33–49.

Mulligan, J., Prescott, A., & Mitchelmore, M. (2004). Students' development of structure in early mathematics. In M. J. Hoines & A. B. Fugelstad (Eds), *Proceedings of the 28th PME international conference* (Vol. 3, pp. 393–400). Bergen: PME.

Pesek, D. D., & Kirshner, D. (2000). Interference of instrumental instruction in subsequent relational learning. *Journal for Research in Mathematics Education, 31*, 524–540.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 157–223). Reston, VA: National Council of Teachers of Mathematics.

Skemp, R. (2006). Relational understanding and instrumental understanding. *Mathematics Teaching in the Middle School, 12*, 88–95.

Whitenack, J., & Yackel, E. (2002). Making mathematical arguments in the primary grades: The importance of explaining and justifying ideas. *Teaching Students Mathematics*, 8, 524–527.

Young-Loveridge, J. (2004). *Patterns of performance and progress on the Numeracy Projects 2001–2003: Further analysis of the Numeracy Project data.* Wellington: Ministry of Education.

Young-Loveridge, J. (2005). Patterns of performance and progress: Analysis of 2004 data. In *Findings from the New Zealand Numeracy Development Project 2004* (pp. 5–20, 115–129). Wellington: Ministry of Education.

Young-Loveridge, J. (2006). Patterns of performance and progress on the Numeracy Development Project: Looking back from 2005. In *Findings from the New Zealand Numeracy Development Projects 2005* (pp. 6–21, 137–155). Wellington: Learning Media.

Young-Loveridge, J. (2007). Patterns of performance and progress on the Numeracy Development Projects: Findings from 2006 for years 5–9 students. In *Findings from the New Zealand Numeracy Development Projects 2006* (pp. 16–32, 154–177). Wellington: Learning Media.

Young-Loveridge, J. (2008). Analysis of 2007 data from the Numeracy Development Projects: What does the picture show? In *Findings from the New Zealand Numeracy Development Projects 2007* (pp. 18–28, 191–211). Wellington: Learning Media.

Young-Loveridge, J. (2009). Patterns of performance and progress of NDP students in 2008. In *Findings from the New Zealand Numeracy Development Projects 2008* (pp. 12–26, 169–184). Wellington: Learning Media.

Young-Loveridge, J., & Mills, J. (2009a). Teaching multi-digit multiplication using array-based materials. In R. Hunter, B. Bicknell, & T. Burgess (Eds), *Crossing Divides*. (Proceedings of the annual conference of the Mathematics Research Group of Australasia, 5–9 July, Wellington, pp. 635–642.)

Young-Loveridge, J., & Mills, J. (2009b). Supporting multiplicative thinking: Multi-digit multiplication using array-based materials. In *Findings from the New Zealand Numeracy Development Projects 2008* (pp. 48–64). Wellington: Learning Media.