Analysis of the Number Framework

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This paper reports on a Rasch analysis of the Number Framework. The analysis provides information on the relative difficulties of the stages and domains of the Framework and the relative extent of the stages within each domain. The analysis supports the validity of the Framework and the theory on which it was based. The stages within each domain describe increasing levels of difficulty, and the difficulty level of each stage across the domains is relatively consistent. While generally supporting the structure of the Framework, the analysis identifies several areas in which it requires refinement.

Background

The Number Framework is one of the key features of the Numeracy Development Projects (NDP). It describes a progression of student learning in terms of both the techniques that students need to develop in order to solve number problems (strategies) and the key items of number knowledge that they need to learn. The Framework is structured in domains, with each domain describing one aspect of students' ability with number. Each domain consists of a number of stages through which students progress as they gain competence (Ministry of Education, 2007).

The Number Framework was developed on the basis of research that investigated children's arithmetical thinking and found identifiable common progressions in the development of number concepts (Bobis et al., 2005). The development of the Framework was theoretically derived and aimed to ensure that each stage was distinct and the stages within each domain described increasingly sophisticated thinking. For example, stage 5 in the multiplicative domain (using known number facts to solve multiplication problems) involves thinking that is theoretically more sophisticated than the thinking in stage 4 in the same domain (using counting strategies). The Number Framework was also developed to ensure that students' stages across the domains are generally well matched. For example, a student at stage 4 on the additive domain (counting on to solve addition problems) is also theoretically likely to be at stage 4 on the multiplicative domain (skip-counting to solve multiplication problems) because both of these stages involve the use of sophisticated counting strategies. To maintain students' consistency across the Framework, some stages in some domains span more than one stage, for example, stages 2–4 in the proportional domain, and some domains omit the lowest stages. For example, stages 2–3 is the first stage of the multiplicative domain.

Limited empirical analysis of the Number Framework was carried out some years ago (Irwin, 2003; Irwin & Niederer, 2002), although the Framework has been updated since that time. This empirical analysis, along with qualitative feedback from the implementation of the NDP, suggested that the difficulty levels of the stages are not equally spaced across any domain and any one stage varies in difficulty across the domains (Irwin, 2003; Irwin & Niederer, 2002; Ward & Thomas, 2002). These factors have limited efforts to compare the progress of samples of students across a range of year levels and Framework stages (Ward & Thomas, 2008; Ward & Thomas, 2009).

This paper reports on an extensive empirical analysis that provides information on the relative difficulties of the domains and the extent of the stages within each domain. A one-parameter logistic (Rasch) item response model is used to place all the stages and domains of the Number Framework on a unidimensional scale. This scale, which is measured in units called logits (natural logarithm units), is used to describe the relative difficulties of the stages within and across each of the domains. It is important to be clear that the unidimensionality of the logit scale constructed under Rasch analysis does not imply that the scale measures a single psychological construct or a single skill. What it does mean, in the case of the Framework, is that, to the extent that the domains fit the scale, a similar cluster of cognitive characteristics and skills is likely to underpin performance across the various domains. For this reason, if the domains fit the scale well, progression through the stages of one domain would tend to be quite highly correlated with progression through the other domains.

Results are presented and discussed in three sections. The first section investigates whether the use of Rasch techniques is justified by the given data set. The second section describes the relative difficulties of the stages and domains of the Number Framework, using the estimated difficulty scores (in logits), and the final section uses transition points between adjacent stages to describe the relative extents of the stages within each domain.

Method

Participants

This paper analyses the results of 3742 students from the 2006 online numeracy database. These students were selected to be part of a multi-year dataset on the basis that it was possible to link their results over three consecutive years, from 2006 to 2008. The dataset was selected to ensure that each student had an entry for the additive strategy domain and for at least one of the other two strategy domains (multiplicative and proportional). Students were also required to have entries for the knowledge domains of both place value and basic facts because these were considered to be the most significant of the knowledge domains in terms of determining students' progress. Further analysis of the results of these students over consecutive years can be found in this volume.

Tables 1 and 2 provide demographic information on the students in the sample.

Table 1
Ethnicity and Gender of Participating Students

Ethnicity	Male (%)	Female (%)	Total
NZ European	50.8	49.2	2625
Māori	50.0	50.0	578
Pasifika	43.1	56.9	232
Asian	45.8	54.2	179
Other	52.3	47.7	128
All	50.0	50.0	3742

Table 2
Year Level and School Decile of Participating Students

Year level	Decile 1–3 (%)	Decile 4–7 (%)	Decile 8–10 (%)	Total
Years 0–1	9.1	25.6	65.3	582
Year 2	9.8	29.6	60.7	737
Year 3	11.9	31.6	56.5	798
Year 4	11.3	29.3	59.4	860
Year 5	15.2	26.8	58.0	343
Year 6+	14.6	27.7	57.7	383
All	11.5	28.9	59.7	3703

Note: 39 students were at schools with no decile rating (private schools).

Method

Student achievement data was collected by teachers who used the Numeracy Project Assessment (NumPA) and entered the data into the online numeracy database. The NumPA is an individual interview that provides information on students' knowledge and strategy stages. The stages against which students are assessed in the NumPA (Ministry of Education, 2008) predominantly correspond with those described by the Number Framework (Ministry of Education, 2007). Where there are discrepancies between the NumPA and the Framework, the stages of the NumPA are used in this paper.

Analysis

Initial analysis investigated whether the use of the Rasch model was appropriate. In particular, the Rasch model fits data to a unidimensional logistic scale, and it is important to ensure that the covariance structure of the data is approximately unidimensional. Correlations between the strategy and knowledge stage ratings of all students (Spearman's coefficients) were investigated and a principal components analysis (without rotation) was used to determine whether a logistic unidimensional scale would be a valid representation of the data.

Having established the approximate unidimensionality of the covariance matrix, difficulty estimates for each stage of each domain were calculated under a Rasch model, using a maximum-log-likelihood procedure. The Rasch model is called a one-parameter model because it produces a single model parameter for each item, which is an estimate of the difficulty of the item in terms of a position on the overall logistic scale. In fact, however, the model also produces an estimate of the ability of each student, again with respect to their position on the overall scale. Item difficulty and student ability are estimated on the same scale, and the difficulty of a stage is defined to be the ability level at which a student would have a 0.5 probability of "correct performance" on that item, that is, of being at that stage or higher. The logit scale constructed by the Rasch analysis is unidimensional, so the difficulty and range of the stages can be meaningfully compared across domains.

Probability functions describing the likelihood of students of any given ability being rated at each stage of each domain were constructed. This enabled the points at which there was an equal probability of being in either of two adjacent stages to be identified. These were characterised as transition points between the stages. These transition points were then used to determine the relative extent of stages on the overall scale.

Findings and Discussion

Factor Structure of the Domains

The Rasch model fits all of the domains to a unidimensional logistic scale. A consequence of this is that students' progression in all of the domains of the Number Framework is likely to be governed by one common set of cognitive processes or to reflect the nature and sequence of instructional programmes. Correlations between students' ratings in the three strategy and two knowledge domains, shown in Table 3, tend to support this, with medium to high correlations evident between all five domains.

Table 3
Correlations between Students' Stage Ratings in the Five Domains

	Additive	Multiplicative	Proportional	Place value	Basic facts
Additive	1	0.665	0.560	0.770	0.807
Multiplicative		1	0.633	0.585	0.651
Proportional			1	0.514	0.598
Place value				1	0.781
Basic facts					1

A principal-components analysis without rotation was conducted to test whether a single dominant component could explain the majority of the variance between students' ratings in the five domains. Table 4 shows the eigenvalues for, and the proportion of the total variance explained by, each component in descending order of magnitude.

Table 4
Eigenvalues of and Variance Explained by Each of the Principle Components

Component	Eigenvalues	Percentage of variance explained	Cumulative percentage
1	3.32	66.4	66.4
2	0.54	10.8	77.2
3	0.43	8.5	85.7
4	0.42	8.4	94.1
5	0.30	6.0	100.0

The finding of a single eigenvalue greater than 1, compared with much smaller eigenvalues for all other components, indicates there is one dominant underlying principal component that explains two-thirds of the total variance. This result confirms that the five domains can be meaningfully fitted to a unidimensional scale and therefore affirms the validity of the Rasch model in this instance. The second component, which explains approximately 11% of the variance, is also of interest because of the way in which the strategy and knowledge domains differentially load onto it. The loadings of each domain on each component are shown in Table 5.

Table 5
Estimated Loadings of Each Domain on Each of the Principal Components

	Component				
Domain	1	2	3	4	5
Additive	0.812	–0.297	0.454	0.065	0.208
Multiplicative	0.869	–0.163	0.002	0.092	–0.457
Proportional	0.817	–0.223	–0.461	0.165	0.205
Place value	0.758	0.597	0.066	0.249	0.048
Basic facts	0.813	0.138	–0.053	–0.563	0.031

All domains load highly on the first component, whereas the second component appears to separate the strategy and knowledge domains. The three strategy domains all load negatively on this component, and the two knowledge domains both load positively. This, to some extent, supports the theoretical distinction between these two sets of domains.

Relative Difficulty of the Stages and Domains

Difficulty estimates for each stage of each domain represent the point on the scale at which a student whose ability is equal to the difficulty of the stage has a 50% chance of being rated at that stage or higher. The probability of students being at the lowest stage or higher in any domain is 100%, so no difficulty estimates can be calculated for the lowest stage in each domain, that is, stage 0 on the additive domain, stages 2–3 in the multiplicative domain, stage 1 in the proportional domain, and stages 0–1 in the place value and basic facts domains. It should also be noted that the difficulty estimates at the extreme ends of the scale, in particular at the upper end, are not as reliable as those nearer the middle. This is because the accuracy of each estimate is related to the number of students at each stage and the sample was weighted to the bottom and middle of the scale. Table 6 shows the proportion of students at each stage of each domain.

Table 6
Proportion of Students Rated at Each Stage of Each Domain

	Domain				
Stage	Additive	Multiplicative	Proportional	Place value	Basic facts
0			16.2	1.0	37.9
1	3.3				
2	20.2			41.5	32.2
3	11.7		72.9*	3.4	4.7
4	37.2	28.4		45.4	39.7
5	24.1	63.0	24.4	8.4	19.0
6	3.3	27.8	2.4	1.1	4.1
7	0.03	8.6	0.2	0.1	0.2
8	0.6	0.1	0.03	0.05	
No. of student results	2919	1333	1474	2890	2116

*This stage in the proportional domain is labelled "stages 2–4".

The difficulty parameter estimates for each stage within each of the five domains are shown in Figure 1. These parameters are estimated with respect to one overall scale, which is measured in logits.

The scale extends from stage 1 in the additive domain, which has a logit value of –6.13, to stage 8 of the place value domain, at 8.72 logits. Exact values for the difficulty estimates can be found in Appendix F.

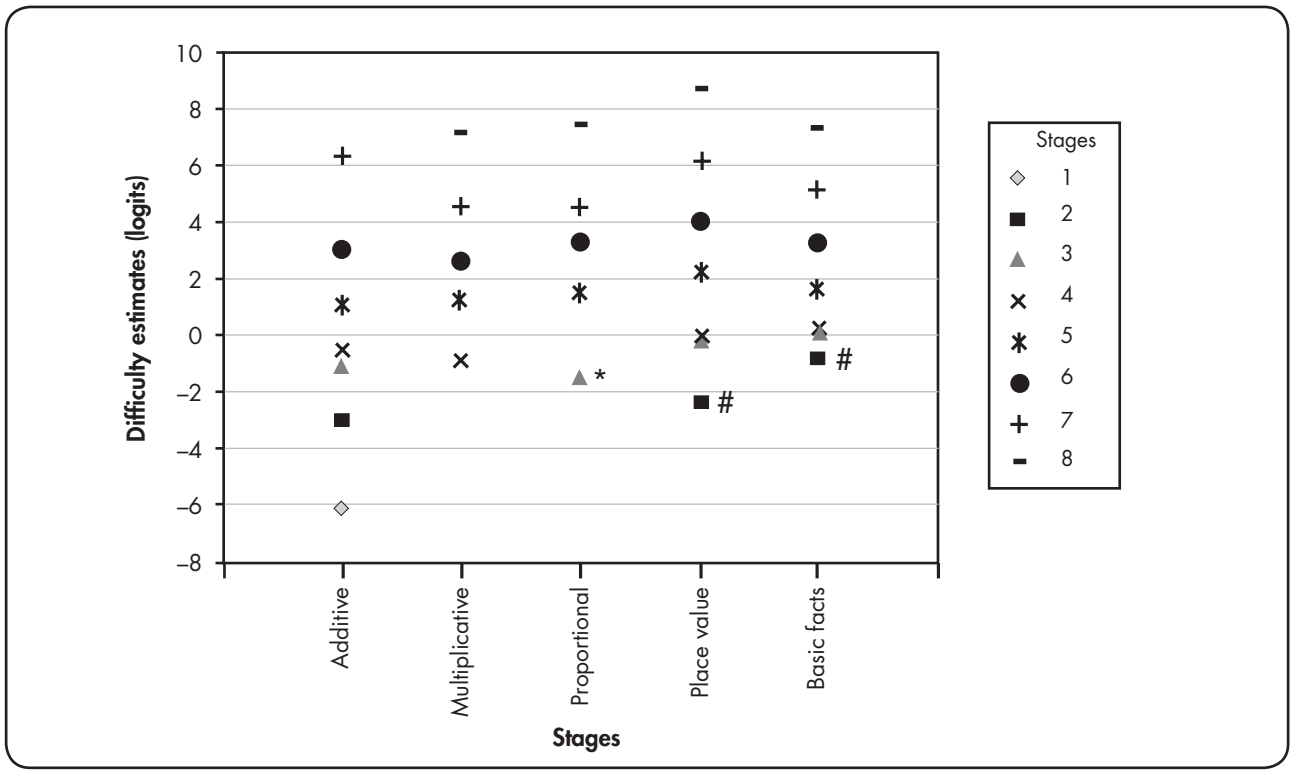

Figure 1. Estimated difficulty scores in logits for each stage of each domain

* This stage would be more accurately labelled as stages 2–4 (proportional domain).

\# These stages would be more accurately labelled as stages 0–1 (place value and basic facts domains).

The analysis provides evidence that the structure of the Number Framework is generally sound. That is, consecutive stages generally increase in difficulty within any domain, there is some consistency between the difficulties of any particular stage across the domains, and most stages of most domains cover a reasonable interval on the scale. Discrepancies in these general patterns are found in stages 3 and 4, particularly in the additive, place value, and basic facts domains. These discrepancies are discussed in detail in the following section.

Of particular interest is the extent to which each stage varies in difficulty across the domains of the Number Framework. Table 7 shows the difficulty range of stages 2–8 across the three strategy and two knowledge domains.

Table 7
Difficulty Range within Each Stage

Stage	Lowest logit estimate	Highest logit estimate	Difficulty range / logits
2	–3.00 (Additive)	–0.83 (Basic facts)	2.17
3	–1.48 (Proportional)	0.11 (Basic facts)	1.59
4	–0.94 (Multiplicative)	0.23 (Basic facts)	1.17
5	1.02 (Additive)	2.20 (Place value)	1.18
6	2.60 (Multiplicative)	3.98 (Place value)	1.38
7	4.50 (Proportional)	6.30 (Additive)	1.80
8	7.17 (Multiplicative)	8.72 (Place value)	1.55

In general, the stages are reasonably consistent in terms of difficulty across the domains, with a difficulty range of between 1.17 (stage 4) to 2.17 logits (stage 2). This result supports, to some extent, the theoretical relationships between the different domains of each stage of the Number Framework. Tables 8 and 9 provide short descriptions of stages 4 and 5. These are the two stages for which there is the most consistent difficulty across the domains.

Table 8
Relative Difficulty of Stage 4 across the Domains

Logit estimate	Domain	Description
0.23	Basic facts	Recalls doubles and teens facts
–0.09	Place value	Counts in tens
–0.53	Additive	Counts on to solve addition and subtraction problems
–0.94	Multiplicative	Skip-counts to solve multiplication and division problems

Within stage 4, basic facts is the most difficult domain and involves students recalling doubles and teens facts. The multiplicative domain is the least difficult at stage 4, with students skip-counting to solve addition and subtraction problems at this stage.

Table 9
Relative Difficulty of Stage 5 across the Domains

Logit estimate	Domain	Description
2.20	Place value	Knows tens in numbers to 1000 and tenths among whole numbers
1.61	Basic facts	Knows addition facts and multiplication facts for 2, 5, and 10
1.47	Proportional	Uses addition facts to solve proportional problems
1.29	Multiplicative	Uses repeated addition or known multiplication facts to solve multiplication and division problems
1.02	Additive	Uses partitioning strategies to solve addition and subtraction problems

Place value is the most difficult domain in stage 5, with students needing to know tens in numbers to 1000 and the number of tenths in whole numbers at this stage. The least difficult domain at stage 5 is the additive domain. This involves students using partitioning strategies to solve addition and subtraction problems.

Stage 2 is the least consistent in terms of difficulty across the domains. Table 10 provides brief descriptions of stage 2 within each of the three domains for which it is relevant.

Table 10
Relative Difficulty of Stage 2 across the Domains

Logit estimate	Domain	Description
–0.83	Basic facts	Instantly recalls facts to 5
–2.38	Place value	Counts in ones
–3.00	Additive	Counts from 1 on materials to solve addition and subtraction problems

Basic facts is the most difficult domain at stage 2, with students recalling addition facts to 5 at this stage. The least difficult domain at stage 2 is the additive domain, with students counting from one on materials to solve addition and subtraction problems.

Focusing on the lower end of the Number Framework, Table 11 shows the ten least difficult stages (excluding the lowest stage) across the five domains. Short descriptions of each of the stages are also included.

Table 11
Stages with the Least Difficulty across All Domains

Logit estimate	Domain	Description
–6.13	Additive, 1	One-to-one counting
–3.00	Additive, 2	Counts from one on materials to solve addition and subtraction problems
–2.38	Place value, 2	Counts in ones
–1.48	Proportional, 2–4	Shares objects equally to identify fractional amounts
–1.06	Additive, 3	Counts from one by imaging to solve addition and subtraction problems
–0.94	Multiplicative, 4	Skip-counting to solve multiplication and division problems
–0.83	Basic facts, 2	Instantly recalls facts to five
–0.53	Additive, 4	Counts on to solve addition and subtraction problems
–0.22	Place value, 3	Counts in fives
–0.09	Place value, 4	Counts in tens

The additive and place value domains dominate the lower end of the Number Framework, with four of the ten least difficult stages of the Framework placed in the additive domain and three in the place value domain. This reflects the content of these domains, both of which include stages focused on counting at the lower end (additive stage 1 and place value stage 2), while other domains start with more difficult skills, for example, recalling facts to five (basic facts domain). One interesting result is the large difference in the difficulty of these two stages, which are both based on students' ability to count. Stage 1 in the additive domain has a difficulty of –6.13 logits, 3.75 logits easier than stage 2 in the place value domain, which has a difficulty of –2.38. One possible explanation for this result is the nature of the tasks used to assess a student's stage within the NumPA. Students need to be able to form a set of eight counters to be rated at stage 1 on the additive domain, but they need to be able to count a given set of 14 dots to be rated at stage 2 on the place value domain. The later task could be regarded as more difficult, even though the descriptors for both stages identify students as counting one-to-one.

At the top end of the Number Framework, six stages have a difficulty estimate that is greater than six logits. These stages are shown in Table 12, along with brief descriptions of each stage.

Table 12
Stages with the Greatest Difficulty across All Domains

Logit estimate	Domain and stage	Description
8.72	Place value, 8	Knows hundredths, names decimals between decimals, converts percentages to decimals, and vice versa
7.45	Proportional, 8	Uses advanced strategies for solving proportional problems
7.33	Basic facts, 8	Knows common factors and multiples
7.17	Multiplicative, 8	Uses advanced strategies to solve multiplication and division problems involving decimals and fractions
6.30	Additive, 7	Uses advanced strategies to solve addition and subtraction problems involving decimals and fractions
6.14	Place value, 7	Knows tenths, orders decimals

The stages that involve using advanced strategies to solve number problems are clearly some of the most difficult stages of the Number Framework. Solving proportional problems is more difficult than solving multiplication and division problems, with relative difficulty scores of 7.45 and 7.17 logits respectively. In turn, solving multiplication and division problems is more difficult than solving addition and subtraction problems, which has a difficulty of 6.30 logits.

The Relative Ranges of Stages and Domains

The probabilistic nature of the Rasch model results in students being described as more or less likely to be at particular stages of the Number Framework, given their ability score, with any given ability score being related to several possible stage ratings. Probability functions showing the probability of a student being at a particular stage on the additive domain given any estimated ability are presented in Figure 2. The functions for the multiplicative and proportional domains can be found in Appendix G. Note that the *x*-axis in each graph represents both student ability and stage difficulty on the logistic scale. This measure relates to students' stage ratings in all five domains and is a consequence of the unidimensional nature of scales constructed by Rasch analysis.

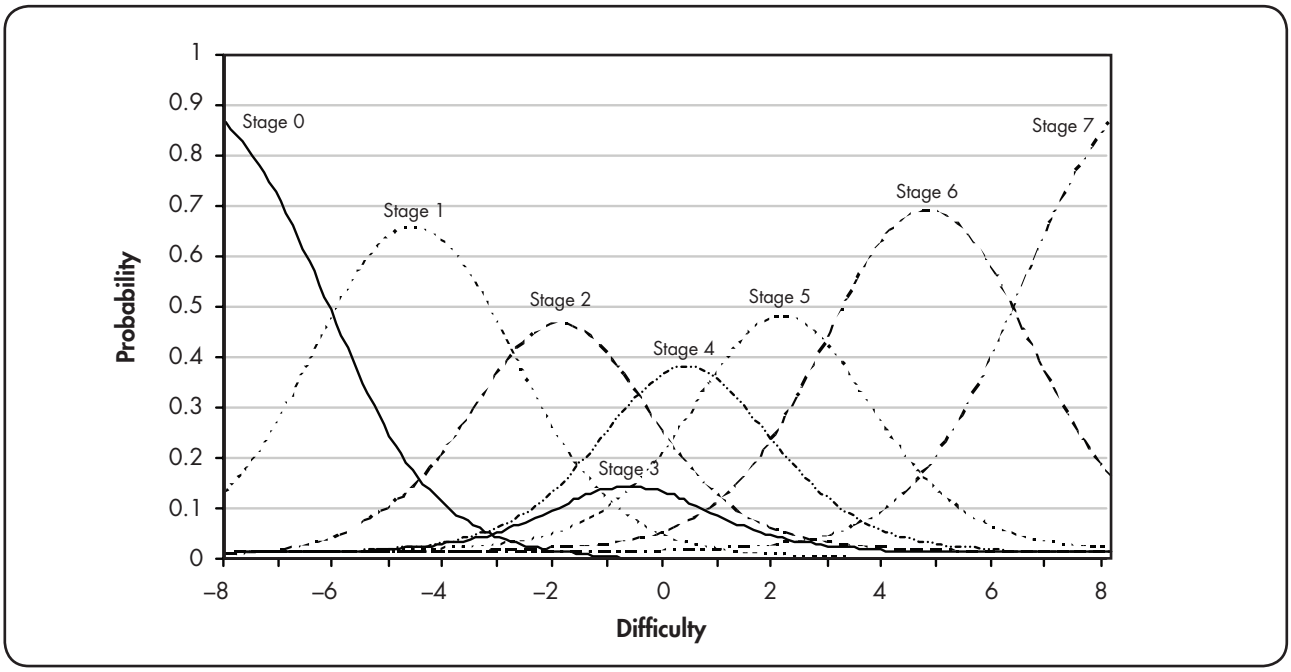

Figure 2. Probability functions for classification at each stage of the additive domain, conditioned on numeracy difficulty

In terms of the additive domain, it seems that there is very little to distinguish stage 3, which requires students to solve addition and subtraction problems by counting from one using imaging, from stage 4, which requires students to count on to solve addition and subtraction problems. There is a difference of just 0.53 logits in difficulty between these stages (Table 15, see Appendix F), and Figure 2 shows that there is never more than a probability of 0.15 that students will be rated at stage 3. This may be because imaging and counting on are so developmentally close that some students either pass very quickly through imaging or omit this stage entirely. An alternative explanation is that imaging is a distinct strategy stage, but it is largely unobservable because it is a cognitive procedure that occurs internally.

Figure 3 shows the probability functions of a student being at a particular stage on the place value domain, given any estimated ability.

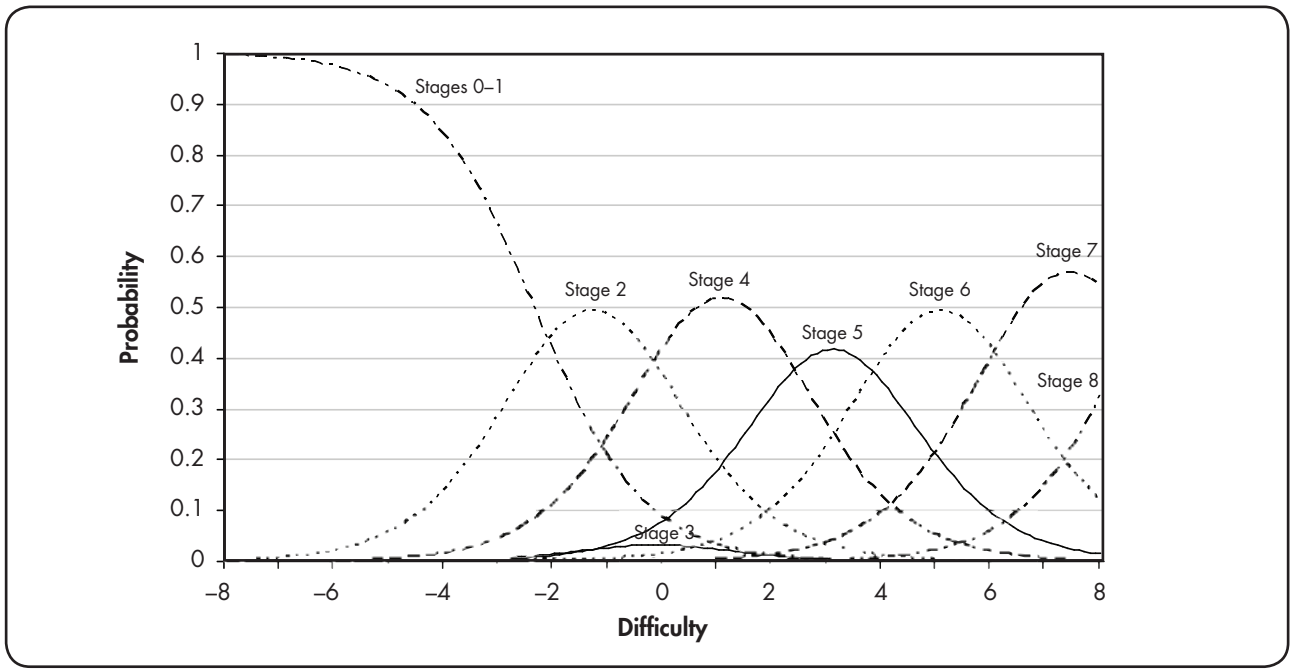

Figure 3. Probability functions for classification at each stage of the place value domain, conditioned on numeracy difficulty

In the place value domain, there is very little difference in terms of difficulty between students' ability to count in fives and ones at stage 3 and their ability to count in tens at stage 4, with just 0.13 logits difference between these stages. Stage 3 is also a very unlikely stage rating, with no ability at which there is more than a 0.05 probability of students being rated at that stage. This may reflect the fact that counting in tens precedes counting in fives in some instructional sequences, or it may be a reflection of the task used to assess knowledge at stages 3 and 4. It could be argued that the assessment task encourages students to count in tens rather than fives because strips of ten dots are consecutively added to a strip of four dots, with students asked to keep a track of how many dots there are in total.

Figure 4 shows the probability functions of a student being at a particular stage on the basic facts domain, given any estimated ability.

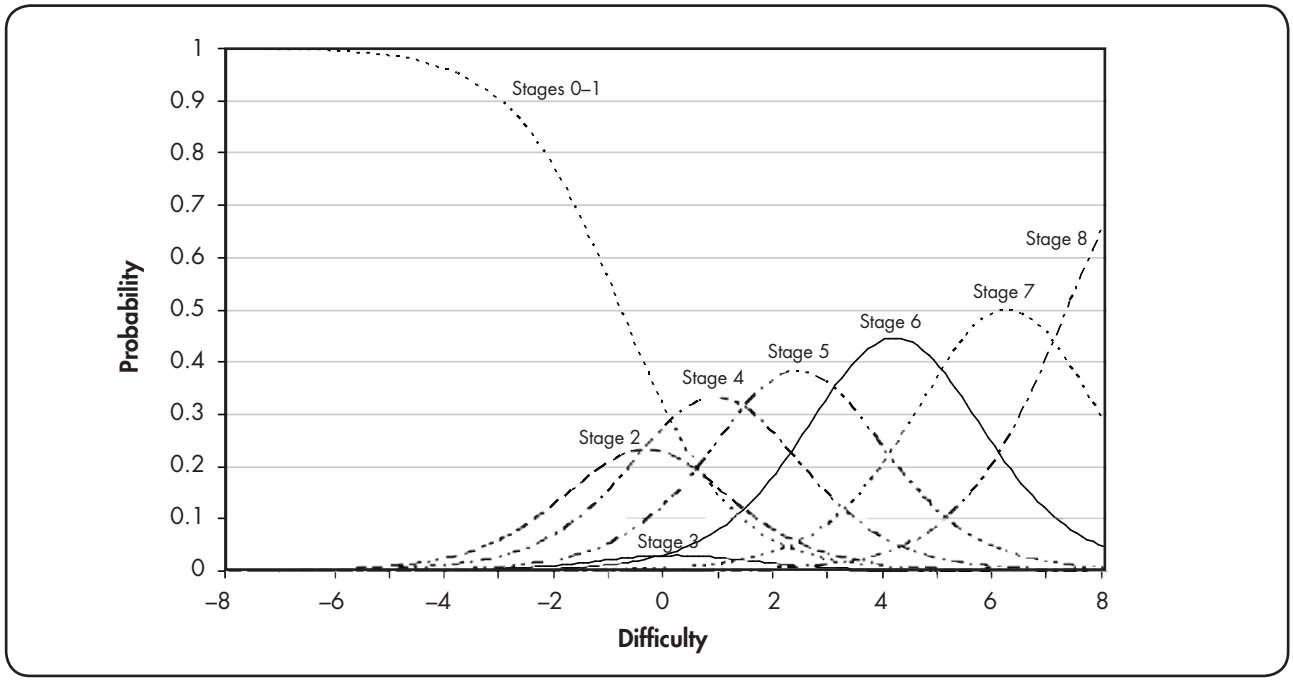

Figure 4. Probability functions for classification at each stage of the basic facts domain, conditioned on numeracy difficulty

The basic facts domain shows a slightly different pattern to that in the additive and place value domains, with three stages close in difficulty level. Stages 3 and 4 are very close in difficulty, with a difference of 0.12 logits between them and never more than a 0.05 probability of students being rated at stage 3. These stages require students to instantly recall facts to ten and to recall doubles and teens facts respectively. It is likely that the closeness of these stages reflects the fact that, in practice, these addition and subtraction facts are all taught together rather than in any structured sequence. Stage 2 in the basic facts domain is also never the most likely stage for students. It has a higher maximum than stage 3, with a maximum probability that students will be rated at stage 2 of just over 0.20. There is also more difference in difficulty between stages 2 and 3, with a difference of 0.94 logits (compared with the 0.12 logits between stages 3 and 4). In practical terms, this means students' ability to instantly recall facts to five (required to be rated at stage 2) is distinct from the highly-related abilities of recalling facts to ten and recalling doubles and teens facts, but overall these three stages are all very closely aligned.

The transition points between consecutive stages were identified as the ability scores at which it is equally likely for a student to be placed on either of two adjacent stages, rounded to the nearest 0.05 logits. For example, Figure 2 shows that students with a numeracy ability of 2.80 logits have a probability of just over 0.40 of being rated at stage 5 on the additive domain and an equal probability of being rated at stage 6 on that domain. Table 13 shows the transition points for the stages of the additive domain; the remaining four domains can be found in Appendix H. Note that stage 3 in the additive domain has been omitted from this analysis because it is not the most likely stage rating for students at any point on the estimated ability scale. Stage 3 in the place value domains and stages 2 and 3 in the basic facts domain were similarly problematic from the point of view of analysing transition points and were therefore omitted from the analysis; transition to the next stage was computed in lieu of the omitted stage.

Table 13
Transition Points on the Additive Domain

Transition between stages					
0–1	1–2	2–4	4–5	5–6	6–7
–6.05	–2.75	–0.75	0.80	2.80	6.20

Figure 5 shows the relative ranges of the stages within the five domains of the Number Framework, calculated using transition points. The highest stage within each domain is not included because the lack of a theoretical end-point to each domain means it cannot be calculated. Note that, because the logit values as estimated are subject to measurement error, the precise values should not be over-interpreted.

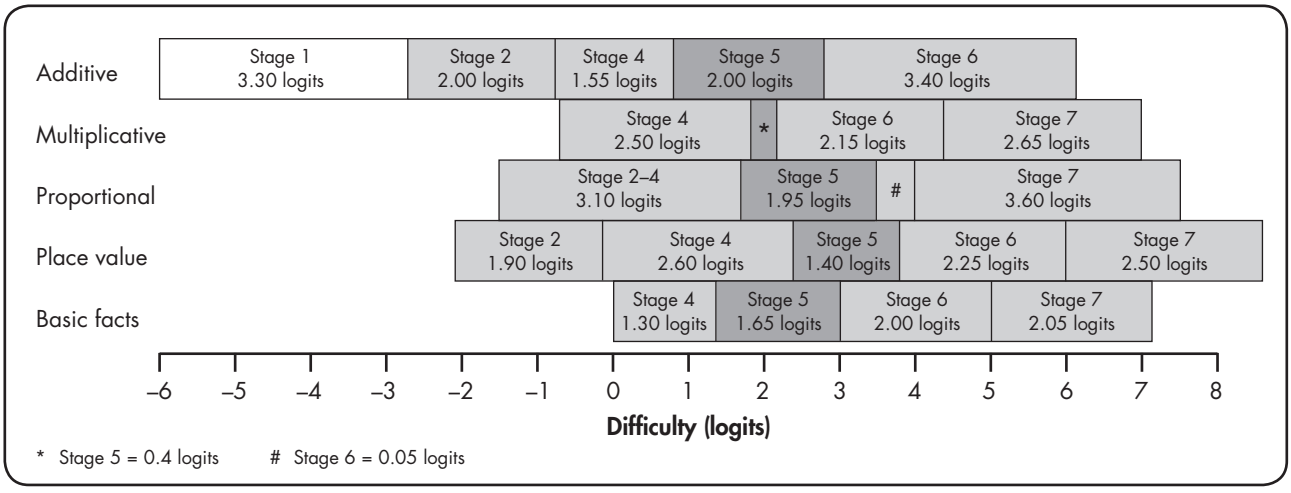

Figure 5. Range of stages in each of the five domains

The ranges of the stages vary from stage 6 in the proportional domain, which has the most limited range (0.05 logits), to stage 7 in the proportional domain, which has the greatest range (3.60 logits). While the difficulty level of each stage varies between the domains, there is no difficulty rating that spans more than two stages across the five domains. This indicates that the stages of the Number Framework are reasonably consistent across the domains. There is a median range of 2.03 logits across the stages and domains shown. It is of note that the stages within each domain with the greatest range shown are those at the highest end of the Framework. Four stages have a range of greater than 3.00 logits across one stage. These stages are described in Table 14.

Table 14
Stages of the Number Framework with Values Greater than 3.0 Logits

Domain and stage	**Description**
Proportional, 7	Uses advanced strategies to solve proportional problems involving ratios, percentages, and fractions
Additive, 6	Uses advanced strategies for the addition and subtraction of whole numbers
Additive, 1	Counts one-to-one
Proportional, 2–4	Shares objects equally to identify fractional amounts

Stage 7 in the proportional domain and stage 6 in the additive domain have two of the greatest ranges shown, and both involve the use of advanced strategies to solve number problems. The range of these stages can most likely be attributed to the variety of strategies that can be used to solve number problems and the complexity of the learning that results. The fact that stage 7 in the multiplicative domain, which requires students to use advanced strategies to solve multiplication and division problems, also has a relatively large range supports this notion. Stage 1 in the additive domain also has a relatively large range, with students needing to master the skill of one-to-one counting to be rated at this stage. The complexity of this stage most likely lies in the fact that students need to co-ordinate knowledge of the number sequence with the skill of one-to-one matching in order to count successfully.

Two stages are particularly limited, with ranges of less than 0.50 logits; these are stage 5 in the multiplicative domain and stage 6 in the proportional domain. The limited ranges of these stages can most likely be attributed to their transitory nature because both represent a partial move towards multiplicative reasoning on the part of the student. Stage 5 in the multiplicative domain requires students to use repeated addition or known multiplication facts to solve multiplication and division problems and can be seen as a transitory step between stage 4, which involves using skip-counting procedures, and stage 6, which involves using only multiplication facts to solve such problems. Stage 6 in the proportional domain has a similar transitory nature. It requires students to use addition and multiplication facts to solve proportional problems, which is a step between stage 5, which involves using addition facts, and stage 7, which involves using multiplication facts to solve proportional problems. Students will quickly limit their use of counting or additive strategies to solve number problems once they see the power of multiplicative relationships, and this accounts for the minimal values of these two transitory stages.

Conclusions

To a large extent, this analysis confirms the validity of the Number Framework and supports the theory on which it was based. The stages within each domain describe increasing levels of difficulty, and the difficulty level of each stage across the domains is relatively consistent.

As well as generally supporting the structure of the Number Framework, the analysis identifies several areas in which the Framework would benefit from refinement. These are:

- Stage 3 in the additive domain, counting from one by imaging to solve addition and subtraction problems, should be removed from the Framework because it is very close in difficulty to stage 4, advanced counting.
- Stages 2, 3, and 4 in the basic facts domain should be combined to include instant recollection of facts to five, facts to ten, and doubles and teens facts in one stage because these are all of similar difficulty.
- Stage 5 in the multiplicative domain and stage 6 in the proportional domain should be removed from the Framework because they are transitory stages that describe the initial development of multiplicative strategies.

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