Students’ Knowledge and Strategies for Solving Equations

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This paper presents results from the second year of an investigation into students’ algebraic thinking. The assessment techniques parallel those used for investigating number in the Numeracy Development Projects. In this study of 621 year 7–10 students, oral interviews with supplementary questions were used to investigate the strategies that students used to solve equations. Basic facts, numeracy strategy stage, and knowledge of aspects of algebra were also assessed. Rasch analysis was used to investigate the difficulty of the equations and student ability, and then the strategies associated with each question were examined. The data provides strong evidence that there is a hierarchy of sophistication of strategies. A large number of the students were unable to solve many of the equations because they were restricted to less sophisticated strategies. Clear relationships were found between the most sophisticated strategy a student used and their numeracy stage, basic facts knowledge, and algebraic knowledge.

Introduction

The success of the Numeracy Development Projects (NDP) in raising New Zealand students’ achievement in number (Thomas & Tagg, 2007; Young-Loveridge, 2007) has prompted initiatives to extend the NDP into early algebra. The NDP are centred on the Number Framework (Ministry of Education, 2003), which describes the progression of students’ arithmetical strategies and the knowledge associated with these strategies. The study reported on here examines students’ strategies for solving linear equations and the relationship of these strategies to numeracy and to algebraic knowledge. The goal is to contribute to a research base that may allow the development of an algebra framework.

Background

Many students struggle with introductory algebra, and teachers have little to guide them in designing programmes of learning. Little is known about the effect of students’ numeracy on the learning of early algebra or about the strategies that students use to solve equations. However, a useful summary of strategies used by students is provided by Kieran (1992), who describes the use of known basic facts, counting techniques, guess-and-check, cover-up, working backwards, and formal operations. In this paper, Kieran’s classification of strategies is modified and extended and “transformations” is the term used to describe Kieran’s “formal operations”.

There is a wealth of research on students’ errors and misconceptions in algebra, some of which was summarised in the 2007 findings from the Secondary Numeracy Project (SNP) (Linsell, 2008). These difficulties include understanding of arithmetical structure, inverse operations, algebraic notation and conventions, operating on unknowns, lack of closure, the equals sign, and treating equations as processes rather than objects. It can be argued that much of the research adopts a deficit model approach, detailing what students can’t do. An alternative approach is to investigate what students actually do when presented with equations to solve and to examine the knowledge and skills associated with the various strategies that different students use.

Teaching how to solve equations has traditionally focused on the type of equations presented to students rather than on the strategies that they are using. If a student is successful at solving a given type of equation, the teacher will often present them with harder equations, irrespective of whether the student is solving by, for example, guess-and-check or working backwards. To move to an approach more consistent with the NDP, in which students are taught according to the highest strategy they have available to them, more information is needed about how students solve equations.

The research questions addressed in this study were:

- What is the relative difficulty of equations?
- What strategies do students use, and is there a hierarchy of strategies?
- What is the impact of context on students' solution strategies?
- What prerequisite knowledge is associated with each strategy?
- What stage of numeracy is associated with each strategy?

Method

This study examined relationships between students' strategies for solving equations and their numeracy stage, basic facts knowledge, algebraic knowledge, and whether the equations were symbolic or in context. A structured diagnostic interview was administered to individual students by the researcher or by the students' classroom teacher. The students' responses were coded and then analysed making use of Rasch analysis (Wright & Masters, 1982). Algebraic knowledge was assessed by a written test, while numeracy stage and basic facts knowledge were assessed through routine procedures in place in the schools. Further details of the assessment tools are provided by Linsell (2008).

Subjects

The study took place in two intermediate schools (years 7 and 8), two high schools (years 9 and 10 only) and one college (years 7–9 only). There was no attempt at representative sampling, but instead the aim was to collect data from a wide range of students. The interview was administered to a total of 621 students in year 7 (n = 196), year 8 (n = 43), year 9 (n = 245), and year 10 (n = 137). Clearly, year 8 students are underrepresented, but this is mitigated by the fact that interviews took place throughout the school year, so students at the beginning of year 9 and the end of year 7 were included. In the two schools in which there was streaming, all classes from each year level were included and in all schools, no students were excluded on the basis of ability.

Diagnostic Interview

The diagnostic interview was developed in a previous study (Linsell, McAusland, Bell, et al., 2006) and was guided by the literature on students' strategies for solving equations (Herscovics & Linchevski, 1994; Kieran, 1992). The interview consisted of a series of increasingly complex equations, which the students were asked to solve along with an explanation of their thinking. The series included 12 pairs of parallel questions: ones that were in context (that is, word problems) and ones that were purely symbolic. The questions were presented on cards so that the more difficult questions could be omitted as required without suggesting to the student that they were not coping. Each question was read to the student to minimise the impact of reading difficulties, including difficulties with reading symbolic equations. Calculators and pencil and paper were available for the students to use, but it was stressed to the students that they could use whatever method they chose. The interviewer recorded what the student did and said and then classified the strategy used according to Table 1. Note that there has been a change in terminology since last year's report (Linsell, 2008): strategy i, previously called formal operations/equation as object, is now called transformations/equation as object. This is in order to clarify that the strategy involves transforming an equation into a new equation one or more times and is not simply the following of some given formal procedure.

Table 1
Classification of Strategies for Solving Equations

Code	Strategy
0	Unable to answer question
a	Known basic facts
b	Counting techniques
c	Inverse operation
d	Guess-and-check
e	Cover up
f	Working backwards, then guess-and-check
g	Working backwards, then known fact
h	Working backwards
i	Transformations/equation as object

Knowledge Test

The assessment of algebraic knowledge was administered as a written test because supplementary questions were not required. The areas investigated in this section were: knowledge of conventions and notation, understanding of the equals sign, understanding of arithmetical structure, understanding of inverse operations, and acceptance of lack of closure.

Numeracy Assessment

All the schools in this study were either NDP or SNP schools and therefore routinely collected numeracy data on their students. In instances where this data was not available, stage of numeracy was assessed using a modified GloSS (Global Strategy Stage) and knowledge of basic facts was assessed using a modified section from NumPA (the Numeracy Project Assessment tool).

Data Analysis

The first stage in the analysis was to determine the difficulty of the equations and the ability of the students. In Rasch models, the probability of a specified response (that is, a right/wrong answer) is modelled as a logistic function of the difference between the person and the item parameter. Before applying this model to the data, it was therefore necessary to ascertain that the variable of item difficulty was unidimensional. Factor analysis was initially employed to verify that a one-factor model was an adequate fit to the data from the strategy interview. Following this, Rasch analysis was used to determine item difficulty and student ability. These scores were then related to the strategies that individual students used for each question.

A proposed hierarchy of strategies was then developed by examining the distributions of ability of students, using each strategy on each question. This permitted each student to be classified according to the most sophisticated strategy they used on any question.

From the numeracy assessment and algebraic knowledge test, each student was assigned a score for numeracy, basic facts, conventions and notation, equivalence, arithmetical structure, inverse operations, and lack of closure. Relationships between each of these measures and the most sophisticated strategy used were then examined.

Results

Item Difficulty

For equations that were presented symbolically, there was a huge variation among the students in the number of equations that they were able to solve, with some questions being much harder than others (see Table 2).

Table 2
Item Difficulty (Symbolic Equations)

Equation	Number of Students with Correct Responses	Percentage of Students with Correct Responses	Rasch Score (Item Difficulty)
$n - 3 = 12$	565	91	–3.750
$18 = 3n$	511	82	–2.910
$n + 46 = 113$	523	84	–3.054
$\frac{n}{20} = 5$	206	33	1.111
$4n + 9 = 37$	400	64	–1.405
$3n - 8 = 19$	382	62	–1.153
$26 = 10 + 4n$	362	58	–0.883
$\frac{n}{4} + 12 = 18$	185	30	1.411
$5n + 70 = 150$	283	46	0.185
$2 + \frac{n}{4} = 8$	153	25	1.873
$5n - 2 = 3n + 6$	109	18	2.581
$2n - 3 = \frac{2n + 17}{5}$	24	4	5.064
Rearrange $v = u + at$	5	1	7.283

In general, one-step equations were easier to solve than two-step equations, which in turn were easier to solve than equations with unknowns on both sides. However, it should also be noted that equations involving division were harder to solve than similar equations with other operators. Nevertheless, one-step equations involving division were easier to solve than two-step equations involving division.

Strategies Used

Two strategies (transformations and guess-and-check) could be used to solve any equation, while others (for example, inverse operation for one-step equations, working backwards for two-step equations) could be used for only a limited number of equations. For every equation (except for the final one) there was a range of strategies successfully used by students, but the distribution of strategies varied from question to question. Responses to three questions are shown in Figure 1 to illustrate the ranges of strategies used.

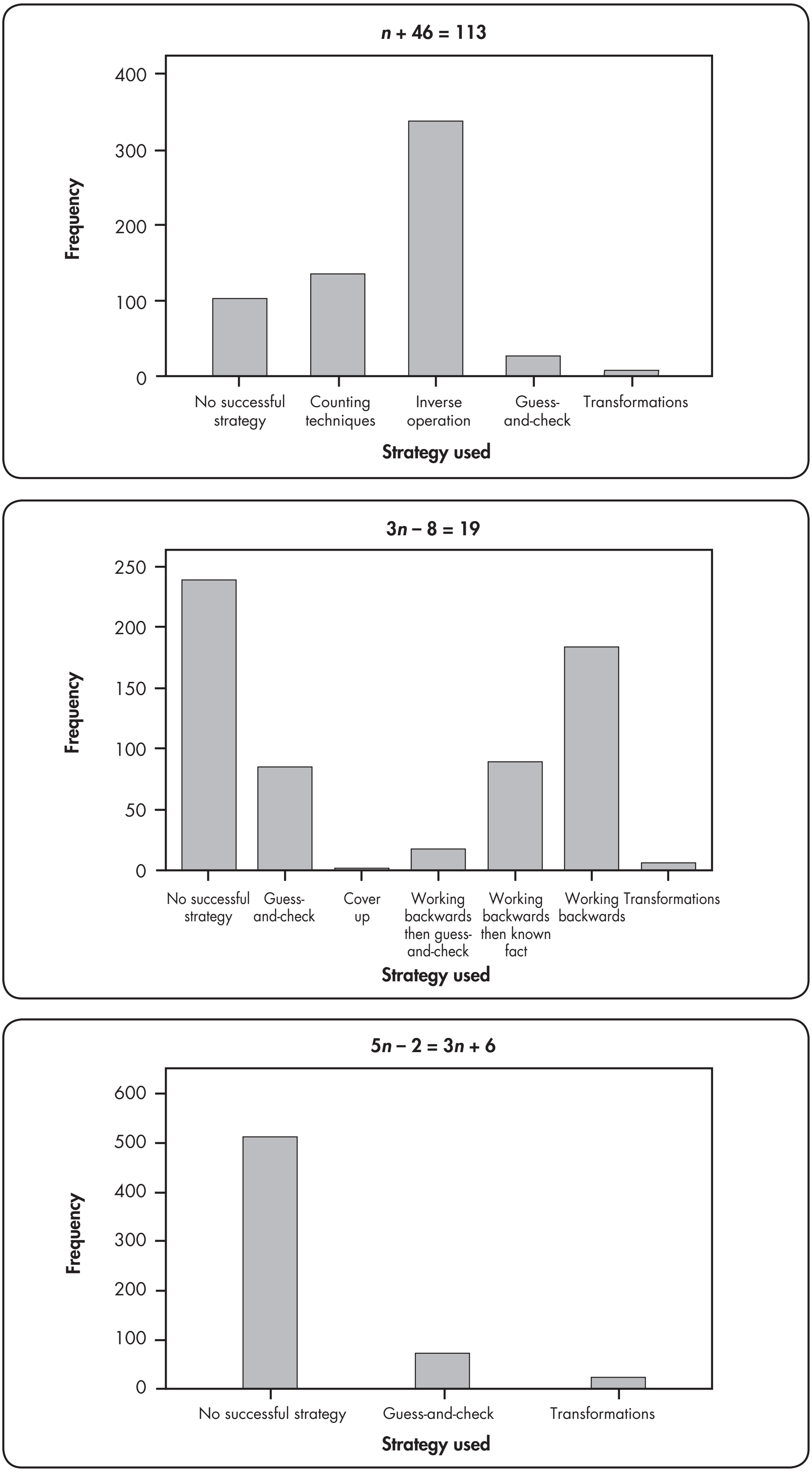
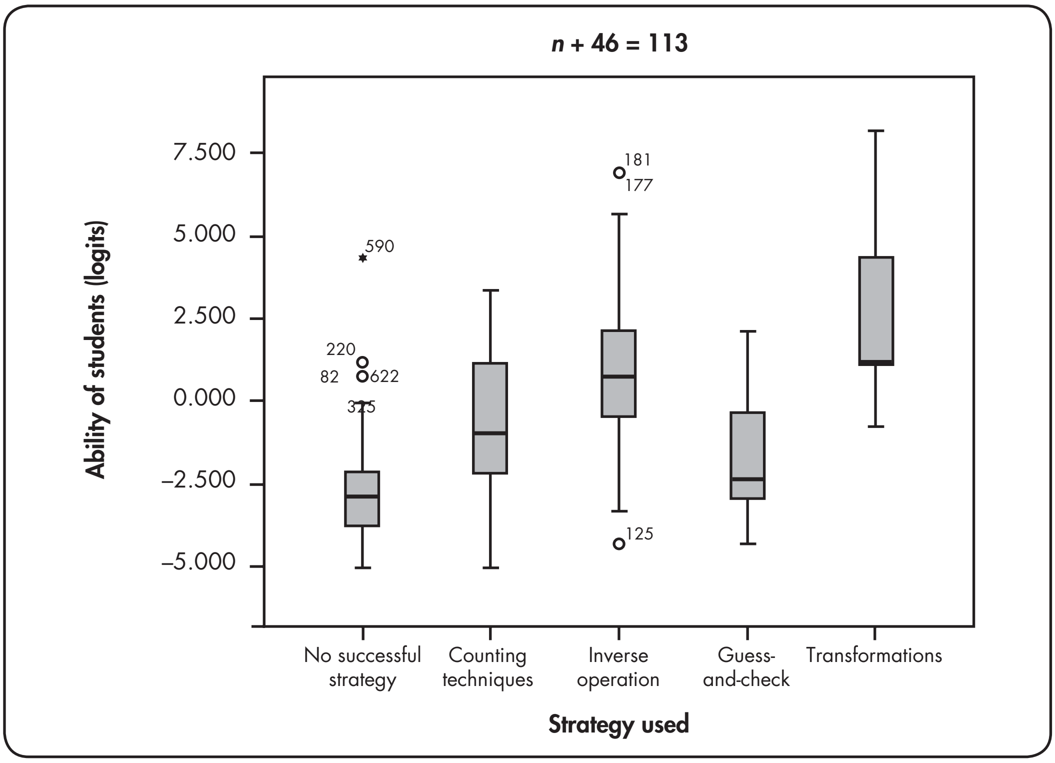


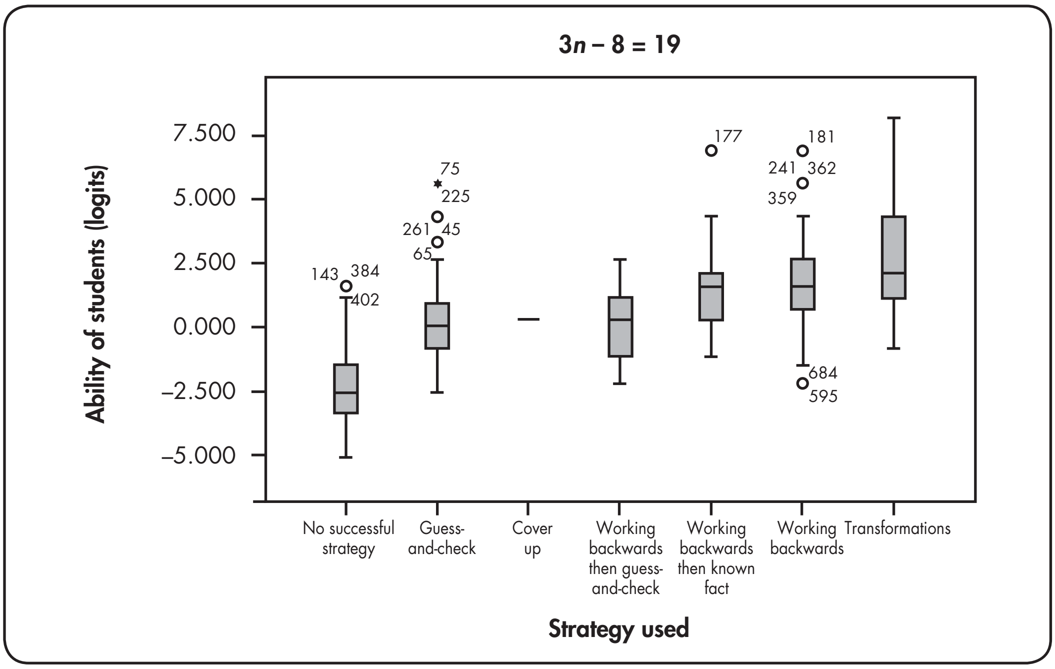
Figure 1. Students' strategies for three equations

Hierarchy of Strategies

To establish a hierarchy of strategies was not straightforward because the pattern of strategy use varied from equation to equation, with some equations lending themselves to being solved by one strategy rather than by another. Another difficulty was that able students often reverted to guess-and-check for difficult questions, even though they used other strategies for easier equations. Less able students, in contrast, used guess-and-check for easy equations and were unable to solve more difficult equations by any strategy.

Therefore, the approach used was to examine the strategies used on a question-by-question basis. For each question, the ability of students using a particular strategy was investigated. Figure 2 shows the results[1] for the same three equations shown in Figure 1.





[1] Rasch analysis calculates item difficulty and student ability on the same scale.

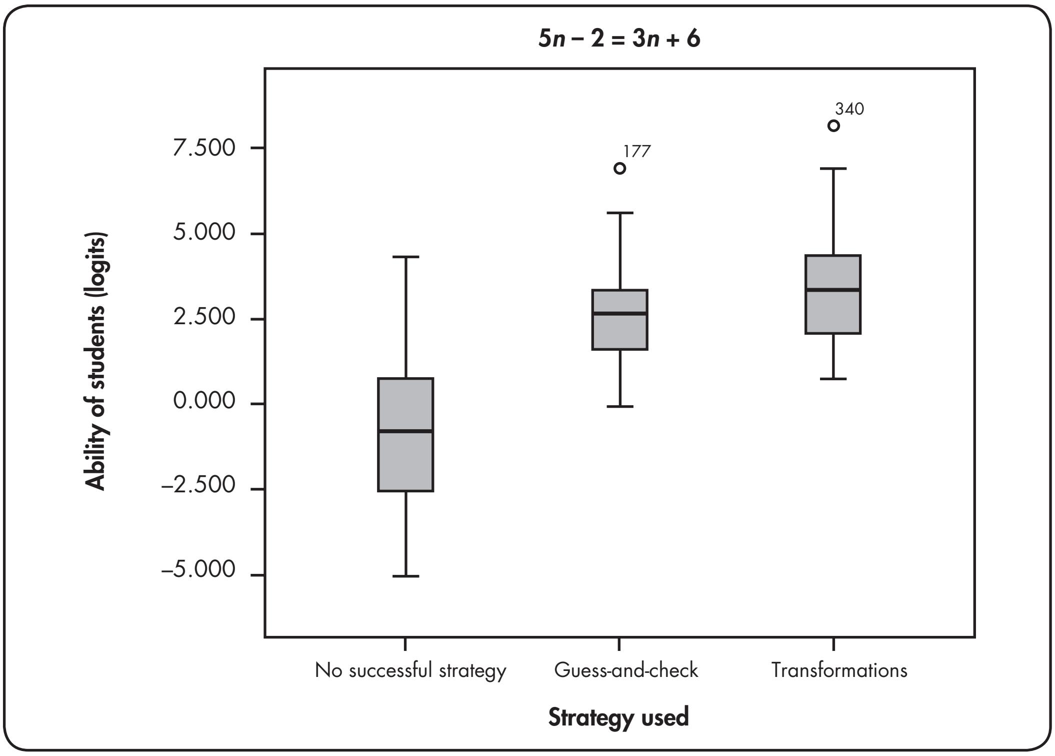


Figure 2. Ability of students using each strategy

For each equation, it was then possible to place the strategies in order according to the mean ability of students using each strategy. For example, the equation $5n - 2 = 3n + 6$ was solved using either guess-and-check or transformations. The mean ability of students using transformations was higher than that of students using guess-and-check, indicating that transformations was the more sophisticated strategy.

The picture that emerged using this approach was fairly consistent, in that the order is the same for nearly all the equations (see Table 3, in conjunction with the explanation of numbers in Table 4).

Table 3
Rank Order of Mean Abilities of Students Using Each Strategy for Each Equation

Strategy	a	b	c	d	e	f	g	h	i
Equation									
$n - 3 = 12$	3		2	4					1
$18 = 3n$	3		2	4					1
$n + 46 = 113$		3	2	4					1
$\frac{n}{20} = 5$	1		3	4					
$4n + 9 = 37$				5	3	6	4	2	1
$3n - 8 = 19$				4		4	3	2	1
$26 = 10 + 4n$	5			6	3	7	4	2	1
$\frac{n + 12}{4} = 18$				5		2	4	3	1
$5n + 70 = 150$				5	6	4	3	2	1
$2 + \frac{n}{4} = 8$				6	5	4	3	2	1
$5n - 2 = 3n + 6$				2					1
$2n - 3 = \frac{2n + 17}{5}$				2					1
$v = u + at$									1

For all equations (except for $\frac{n}{20} = 5$), transformations was the strategy used by the most able students and guess-and-check by the least able. For one-step equations, it was not possible to discern between counting strategies and known basic facts because, although students used a range of strategies over all the equations, for any particular equation this range never included both counting and known basic facts. However, for three of the four one-step equations, inverse operations were used by the more able students rather than by students using either counting strategies or known basic facts. The exception was $\frac{n}{20} = 5$, which was far more difficult than the other one-step equations. For this equation, the most able students solved it using a known basic fact. Cover up was used by such a small number of students that no clear relationship to the other strategies emerged. For five of the six two-step equations, working backwards was used by the more able students rather than by those using working backwards, then known fact, which in turn was used by the more able students rather than by those using working backwards, then guess-and-check. The exception was $\frac{n + 12}{4} = 18$, but the number of students using any strategy other than working backwards was too small to draw any conclusions. The strategies used only on one-step equations clearly could not be compared directly with those used only on two-step equations. However, two-step equations are much harder than one-step, and working backwards involves using inverse operations.

The order of sophistication of strategies indicated by this analysis (see Table 3) is shown in Table 4.

Table 4
Rank Order of Strategies

Rank	Strategy
1	No successful strategy
2	Guess-and-check
3	Counting techniques / Known basic facts
4	Inverse operations
5	Working backwards, then guess-and-check
6	Working backwards, then known fact
7	Working backwards
8	Transformations

The students were then classified according to the most sophisticated strategy they used on any symbolic equation (see Figure 3).

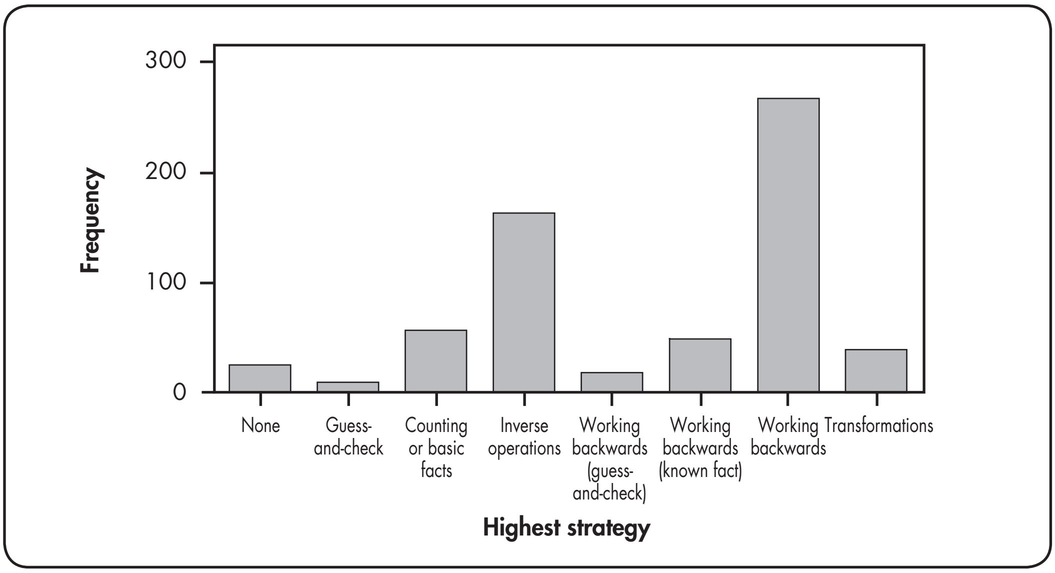


Figure 3. Most sophisticated strategy used on any symbolic equation

The number of students able to solve equations by performing transformations was very low (38). It is also worth noting that a significant number of students (89) were not even able to use inverse operations. Also of note is the number of students (65) who could solve some two-step equations but who were not fully working backwards.

Impact of Context

The picture that emerged in relation to students' solutions of equations that were given to them as word problems in context was very similar to that of symbolic equations. The diagnostic interview included 12 pairs of parallel questions. Table 5 shows the item difficulty of each question. Note that the table shows the structure of the word problems, but the problems read to students did not include these equations. For example, the first problem (shown in the table as $n - 5 = 17$) was "I left home this morning with some money, spent \$5, and have \$17 left. How much did I start with?"

Table 5
Item Difficulty (Rasch Scores)

	Symbolic	In context	Symbolic		In context	
	Equation		% correct	Item difficulty	% correct	Item difficulty
1	$n - 3 = 12$	$n - 5 = 17$	91	–3.75	96	–4.63
2	$18 = 3n$	$24 = 3n$	82	–2.91	67	–1.46
3	$n + 46 = 113$	$n + 24 = 89$	84	–3.05	88	–3.38
4	$\frac{n}{20} = 5$	$\frac{n}{20} = 4$	33	1.11	83	–2.98
5	$4n + 9 = 37$	$4n + 5 = 37$	64	–1.41	75	–2.29
6	$3n - 8 = 19$	$3n - 7 = 17$	62	–1.15	67	–1.55
7	$\frac{n + 12}{4} = 18$	$\frac{n + 11}{4} = 19$	30	1.41	52	–0.31
8	$5n + 70 = 150$	$5n + 70 = 250$	46	0.19	52	–0.34
9	$2 + \frac{n}{4} = 8$	$2 + \frac{n}{4} = 7$	25	1.87	28	1.45
10	$5n - 2 = 3n + 6$	$5n - 3 = 3n + 9$	18	2.58	13	2.89
11	$2n - 3 = \frac{2n + 17}{5}$	$4n - 2 = \frac{5n + 14}{2}$	4	5.06	2	6.26
12	Rearrange $v = u + at$	Rearrange $s = \frac{(v + u)t}{2}$	1	7.28	0	

The harder equations (with unknowns on both sides) were slightly easier when presented symbolically. However, nearly all one-step and two-step equations were easier when presented in context. There was one exception (the second pair of equations) and a few dramatic differences in difficulty that will be commented on when the strategies that students used are examined.

For all of the one-step equations that were in context, compared with their symbolic counterparts, there was a greater use of inverse operations than of counting strategies, known facts, or guess-and-check. This was true even for the second pair of equations, in which the order of difficulty was reversed. The context was "I have 24 CDs. This is three times as many as my brother has. How many CDs does he have?" Many more students than expected got this wrong by multiplying 24 by 3 rather than dividing. It would appear that the words "three times" confused them.

For the fourth, seventh, and ninth pairs of equations, students found the symbolic form much more difficult than one might expect. All these equations involved a division structure. The contexts for the fourth and seventh were: "When I shared a packet of lollies round my class of 20 students, they got 4 each. How many lollies were in the packet?" and "Our kapa haka group is made up of some Māori students and 11 Pākehā students. The whole group is divided into 4 equal-sized groups for practices. Each of the practice groups has 19 students in it. How many Māori students are there in our kapa haka group?" It would appear that students saw these as multiplication rather than division problems and found them much easier than the symbolic equivalents.

For all of the two-step equations, there was a greater proportion of students using the working backwards strategy for equations that were in context compared with those using this strategy for symbolic equations. Conversely, there was a greater proportion of students using less sophisticated strategies for symbolic equations compared with those using these strategies for equations in context. There was also a small number of students who used transformations for two-step symbolic equations.

Prerequisite Skills and Knowledge

The relationships between each student's prerequisite knowledge and skills and the most sophisticated strategy that they were able to employ was then investigated. For these analyses, those students whose most sophisticated strategy was guess-and-check, counting, known basic fact, or no successful strategy were grouped together as primitive strategies. Those students whose most sophisticated strategy was either working backwards, then guess-and-check, or working backwards, then known fact, were grouped together as partially working backwards. The relationship between numeracy strategy stage and highest algebraic strategy is shown in Figure 4.

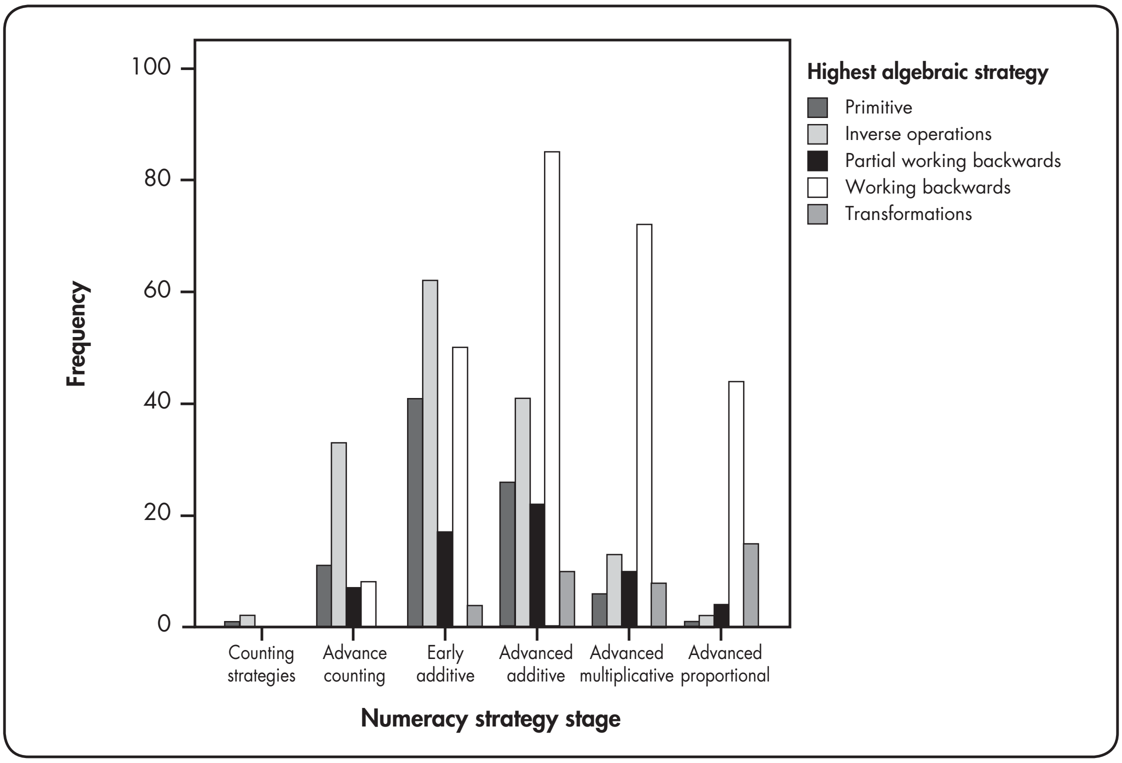


Figure 4. Relationship between students' numeracy and their most sophisticated strategy for solving equations

It is clear that students with poor numeracy skills were largely restricted to less sophisticated strategies. The great majority of students with good numeracy skills were able to use more sophisticated strategies, with very few of them using only less sophisticated strategies. A very similar picture to this emerged for students' knowledge of basic facts. Students with poor knowledge of basic facts were largely restricted to less sophisticated strategies, while most students with good knowledge of basic facts were able to use the more sophisticated strategies.

The relationship between understanding of arithmetical structure and highest algebraic strategy is shown in Figure 5.

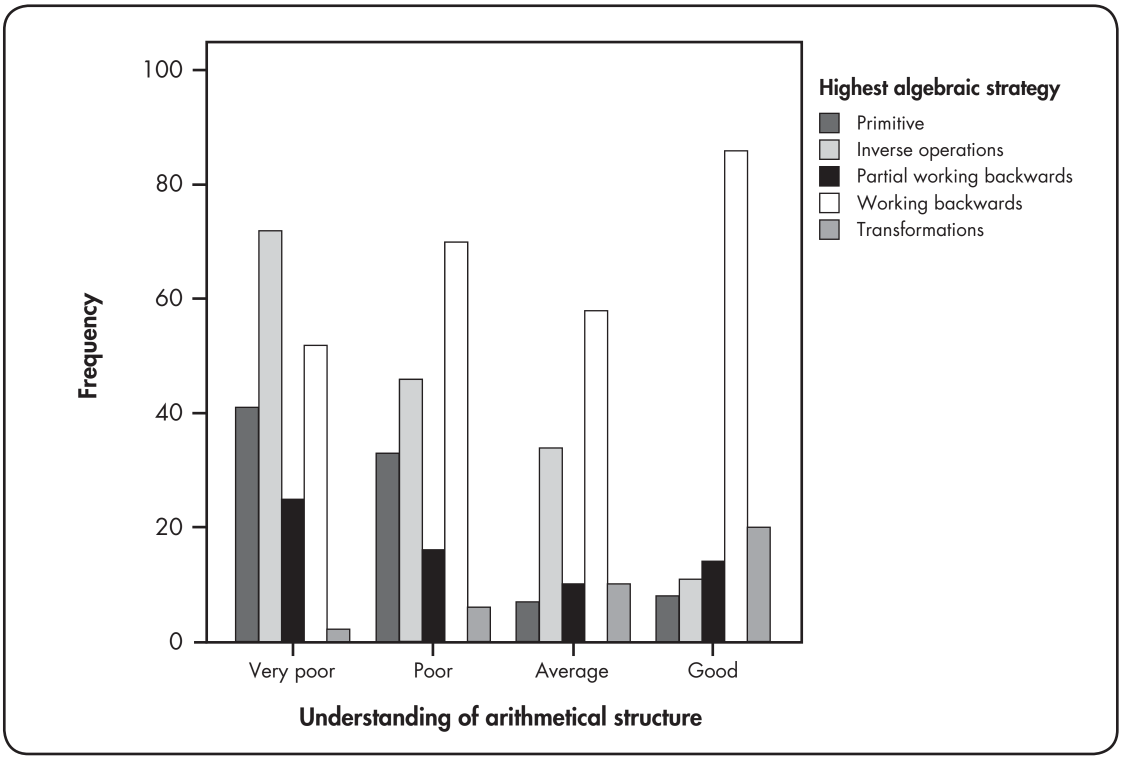


Figure 5. Relationship between students' understanding of arithmetical structure and their most sophisticated strategy for solving equations

Students' understanding of arithmetical structure had a dramatic impact on the most sophisticated strategy they were able to use. As their understanding of arithmetical structure increased, so did their use of more sophisticated algebraic strategies. The pictures for understanding of inverse operations, acceptance of lack of closure, and understanding of equivalence were very similar. The only area of knowledge that did not have such a clear relationship with sophistication of algebraic strategy was algebraic notation and convention. Students with good knowledge in this area used slightly more sophisticated strategies, but the relationship was not so convincing. Of particular note is the impact of students' understanding of equivalence and, to a slightly lesser extent, their acceptance of lack of closure (see Figure 6).

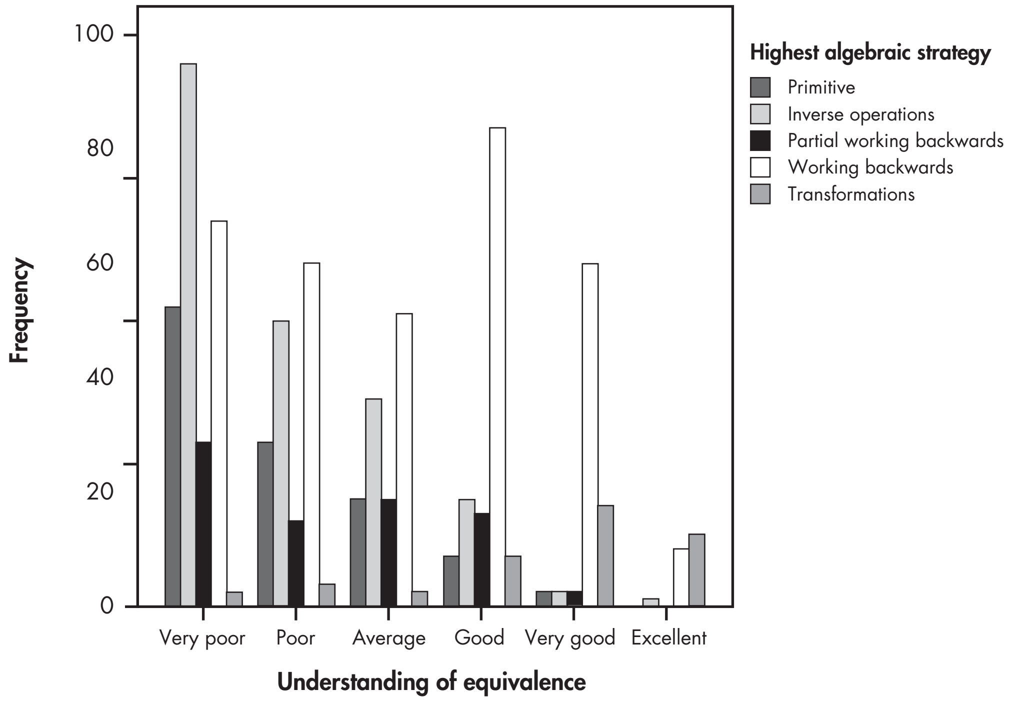


Figure 6. Relationship between students' understanding of equivalence and their most sophisticated strategy for solving equations

As students' understanding of equivalence increased, the ratio of the number of students using transformations to those using working backwards also increased. Of those students with an excellent understanding of equivalence, there was a greater number using transformations than using working backwards. This difference in proportion between those using working backwards and those using transformations was not so marked for the other areas of prerequisite knowledge investigated, although it was also large for acceptance of lack of closure.

Discussion

This study has demonstrated how incredibly difficult some equations are to solve compared with others. It has been known for a long time that students find equations with unknowns on both sides very difficult (Herscovics & Linchevski, 1994), and Sfard (1991) has even suggested that viewing equations as objects may be beyond the grasp of many students. However, this study is the first to compare item difficulty on a Rasch scale. Rasch scores of item difficulty follow an approximately normal distribution, and to find that some items have a difficulty score over two standard deviations above the mean confirms their difficulty. *The New Zealand Curriculum* achievement objectives (Ministry of Education, 2007) give little indication of this difficulty. At level 4, the relevant objective is "Form and solve simple linear equations", and at level 5, it is "Form and solve simple linear and quadratic equations". The level 3 objective of "Record and interpret additive and simple multiplicative strategies, using words, symbols, and diagrams, with an understanding of equality" is useful in focusing on equivalence, but it does not spell out that students need to find unknown values in any position in a statement of equivalence. Educators need to appreciate how huge the range of difficulty is and not trivialise the solving of equations down to a few lessons on specific procedures.

Although equations in context were generally found to be slightly easier than symbolic equations, the general picture for item difficulty was very similar when the two sets of parallel questions were compared. This demonstrates that it is primarily the structure of problems that makes them difficult, not the algebraic symbolism. This finding is backed up by the lack of a strong relationship between students' understanding of algebraic notation and their most sophisticated algebraic strategy. Teaching of algebra should therefore be focusing on the structure of problems. The additional difficulties that students experienced with equations involving division highlight the need for greater emphasis to be placed on a variety of operators within both one-step and two-step equations.

Insights into why students found some equations so difficult to solve were obtained by examining the strategies they employed to solve them. It was clear that students were using a wide variety of strategies to solve all the equations. For one-step equations, many students were getting correct solutions but never used an inverse operation. To attempt to move these students on to two-step equations would be courting disaster. Inverse operations are involved in all the successful strategies for two-step equations other than guess-and-check. Students are therefore likely to be reduced to using guess-and-check or learning a specific procedure that applies only to specific structures. Similarly, many students were obtaining correct solutions to some two-step equations but were only partially using working backwards or were even using guess-and-check. This point, that the strategy of working backwards is less homogeneous than previously reported, is important. Many students are only just grasping the strategy and can use it only when the first step reveals a known basic fact to them for the next step. These students use the strategy of working backwards, then known facts. Other students are prevented from fully using working backwards because of lack of knowledge of multiplication and division facts. These students use the strategy of working backwards, then guess-and-check. To attempt to move these students on to using transformations would almost certainly be premature. When teachers suggest to students that an equation is like a balance pivoted about the equals sign, it is hard to imagine why it would be difficult to do the same thing to both sides of the equation and keep it balanced. However, this study has confirmed just how difficult the strategy of using transformations is for students. Using transformations requires seeing an equation as an object to be acted on (Sfard, 1991), but it is clear that most students see equations as processes.

In order to investigate the relative sophistication of strategies, it was assumed that the more sophisticated strategies would be chosen by the more able students. The findings using this approach were consistent with the fact that there was only one student who was able to perform transformations but could not also work backwards (this low-ability student appeared to be following a procedure of doing the same thing to both sides for simple equations) and that all students who could work backwards could also use inverse operations. This study has shown that the strategy of solving one-step equations by inverse operations is used by the more able students rather than by those who use either known basic facts or counting strategies. These strategies in turn were used by the more able students rather than by those who solved one-step equations using guess-and-check. Similarly, the strategies of solving two-step equations by partially working backwards were used by less able students rather than by those fully working backwards. On any particular equation, students chose a strategy that was sufficient to solve the equation rather than using their most sophisticated strategy. However, it is suggested that the different strategies are not merely a matter of choice but that the most sophisticated strategy that a student ever uses is indicative of conceptual development.

The impact of context on students' ability to solve equivalent problems was very interesting. In general, context problems were found to be easier than symbolic problems until the difficulty level of unknowns on both sides was reached. However, most school programmes focus on teaching skills for solving symbolic equations. Solving word problems is usually regarded as harder and introduced later as an application of these skills. An alternative perspective on contexts is to view them as models of the mathematics. Models are an important feature of Realistic Mathematics Education (RME) (Gravemeijer, 1997). Traditionally, models are derived from formal mathematics, whereas in RME, models are derived from real situations that students have experienced and are chosen to reflect the informal strategies of students. Initially, a model of a situation that is familiar to the students is used. Next, through generalising and formalising, the model becomes an entity in its own right. Finally, it becomes possible to use the model for mathematical reasoning. Gravemeijer describes this as a transition from *model-of* to *model-for*. The nature of a model therefore evolves from being highly context-specific to deriving its meaning from a mathematical framework. In contrast, when pre-existing models are given to students to help them solve problems, the students are expected to use them in prescribed ways that may not be clear to them. The results from this study are consistent with Gravemeijer's perspective and suggest that algebra would be better introduced in context rather than just as symbols.

The impact of context on the strategies that students used may help to explain why students found these problems easier. For one-step equations, there was much higher use of inverse operations than of less sophisticated strategies. It appears likely that contexts allow students to perceive the structure of a problem in more than one way. For example, the problem "When I shared a packet of lollies round my class of 20 students, they got 4 each. How many lollies were in the packet?" has the structure $\frac{n}{20} = 4$, but may be viewed as "The number of lollies is 4 for each of the 20 students", with a structure of $n = 20 \times 4$. The context is therefore naturally leading the student into an inverse operation. If this is the case, then the role of the teacher should be to scaffold the writing of symbolic equations to describe contexts and then to explore and symbolise the solution strategies of the students.

There was a high correspondence between numeracy strategy stage and the most sophisticated strategy a student was able to use to solve equations. Only for students who were at the advanced multiplicative or advanced proportional thinking stages did the majority solve equations by using working backwards or transformations. Students at lower stages of numeracy were largely restricted to less sophisticated strategies. The findings from this study strongly suggest that prerequisite numeracy should be considered when designing teaching programmes for algebra. However, there were students who did not score highly on GloSS but were able to use sophisticated strategies for solving equations. These students were invariably efficient at using algorithms for computations and often came from primary schools that did not promote NDP numeracy. The algebra diagnostic tool may be more useful than GloSS for revealing the thinking of students at the upper end of the Number Framework. This is because GloSS focuses on mental strategies (and does not value the use of algorithms), whereas the algebra tool has a focus on students' understanding of mathematical structure.

There was a very strong relationship between students' knowledge of basic facts and their highest algebraic strategy. Any student who was at stage 6 or below on the Number Framework for basic facts was unlikely to be able to solve equations by working backwards or by transformations. This finding emphasises the critical importance of instant recall of all basic facts, including multiplication and division.

There were also strong associations between students' highest algebraic strategies and their understanding of arithmetical structure, inverse operations, lack of closure, and equivalence. There was not such a strong association between students' highest algebraic strategies and their knowledge of algebraic conventions and notation. The relationship between students' highest algebraic strategy and their understanding of equivalence was particularly interesting. Understanding of equivalence and, to a lesser extent, acceptance of lack of closure had much higher impacts on whether a student could use transformations compared with using the strategy of working backwards than did the other areas of algebraic knowledge. Given the reasonably large number of students who could work backwards and the very small number who could use transformations, these findings may have significant implications for teaching.

Conclusions

Consistent with the perspective of Filloy and Sutherland (1996), it is suggested that the strategies described in this study are not simply alternative approaches to solving equations but represent different stages of conceptual development. Instead of looking at how hard equations are to solve and whether students get them right, it appears to be more useful to look at the strategies that students use. The approach used in this study is very similar to that used in the NDP, with strategy being separated out from the knowledge required for strategy use. This approach allows the classification of the students according to their most sophisticated strategy rather than by the most difficult equation they are able to solve. Within numeracy teaching, students are grouped for instruction according to their most sophisticated strategy. It is suggested that a similar approach to grouping students is likely to be beneficial for teaching students to solve equations.

Also consistent with the NDP, the teaching of prerequisite knowledge needs to be addressed. To solve one-step equations, students need to understand inverse operations and to know their basic facts. To solve two-step equations by working backwards, students also need to understand arithmetical structure. To solve equations by using transformations, students need to understand equivalence and also accept lack of closure.

The third area of findings consistent with the NDP concerns the role of context. Questions that were in context were easier than equivalent symbolic questions. This suggests that a version of the numeracy teaching model should be employed for teaching algebra. We should start with contexts that are meaningful to students, preferably involving concrete materials. Teachers should scaffold students so that they can symbolise the structure of the problems and their solution strategies before expecting them to visualise a concrete representation and finally to operate on abstract symbolic structures.

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