Year 7–8 Students' Solution Strategies for a Task Involving Addition of Unlike Fractions

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This study reports on data gathered from 238 year 7–8 students in six intermediate schools who were given a task involving addition with fractions ($\frac{3}{4}+\frac{7}{8}$). Only 32 students (13%) found a correct answer for the problem, and some of those solved it using procedural knowledge rather than a deep conceptual understanding of fractions. Half of the students gave an incorrect answer. The most common error, shown by almost a quarter of the students, was to add the numerators and/or denominators (the "add across" error). More than a third of the students did not attempt the problem. Students' difficulties are analysed and the implications of the findings for teachers discussed. The potential value of the "make a whole" strategy for helping students understand about the properties of fractional numbers is considered.

Fractional or rational numbers are important in everyday-life situations (Anthony & Walshaw, in press), allowing us to answer questions not just about "how many" but also about "how much". Even preschool children use fractions when trying to determine "fair shares". Real-life measurement problems often require an understanding of rational numbers if precise measurements are to be made. Many everyday activities, such as shopping, rely on an understanding of rates such as price per litre or price per kilogram. Speed limits are presented as the relationship between distance and time (for example, 50 kilometres per hour). Percentages, a particular type of fractional number, are important to anyone with a mortgage or a bank loan. Discounts are usually presented as percentage reductions in price.

Learning about fractions presents considerable challenges for students throughout their school years (Anthony & Walshaw, in press; Behr, Lesh, Post, & Silver, 1983; Brown & Quinn, 2006; Charalambous & Pitta-Pantazi, 2005; Davis, Hunting & Pearn, 1993; Empson, 1999, 2003; Hunting 1994, Lamon, 2007; Pearn & Stephens, 2004; Smith, 2002; Usiskin, 2007; van de Walle, 2004; Verschaffel, Greer, & Torbeyns, 2006). The difficulties that students experience with fractions can cause problems with other domains in mathematics such as algebra, measurement, and ratio and proportion concepts (Behr et al., 1983; Lamon, 2007; van de Walle, 2004). On the other hand, teaching students how to abstract mathematical ideas in the context of fractions can be extremely beneficial to their algebra learning (Wu, 2002).

Although fractions are known to be difficult to teach and learn, they have been described as one of the most "mathematically rich" and "cognitively complicated" areas of primary school mathematics (Smith, 2002). Moreover, they "are among the most complex and important mathematical ideas children encounter during their pre-secondary school years" (Behr et al., 1983, p. 91). It seems likely that the difficulties that students experience with fractions are related to their complexity. Various frameworks have been proposed to account for the different ways that fractions can be interpreted, including Kieren's system of five sub-constructs (see Behr et al., 1983). Understanding fractions requires an understanding of each of the sub-constructs as well as the ways in which the sub-constructs are connected. Arguably the most important sub-construct, the one underpinning all other sub-constructs, is the *part–whole* or *partitioning* sub-construct. However, unlike the partitioning that occurs with the addition and subtraction of whole numbers (which can be of unequal parts), partitioning for fractions (as well as for multiplication and division) must be of equal-sized parts (see Pothier & Sawada, 1983). For this reason, *equivalence* is a key aspect of the part–whole sub-construct.

Kieran's other four fraction sub-constructs include *ratio*, the idea of relative magnitude, necessary for understanding ideas about proportion and equivalence (as in renaming $\frac{3}{4}$ as $\frac{6}{8}$, $\frac{9}{12}$, $\frac{15}{20}$, and so on); *operator*, necessary for the multiplication of fractions (as in $\frac{3}{4}$ of 10 metres); *quotient*, necessary for problem solving (as in $\frac{1}{4}$ of 20 means the division of 20 by 4); and *measure*, necessary for addition of fractions (as in $\frac{3}{4}$ is the same as $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$). Another way of looking at fractions is to see them as both a *process* (for example, $\frac{3}{4}$ involves division of 3 by 4) and a *product* that results from a process ($\frac{3}{4}$ is the result of dividing 3 by 4) (see Verschaffel, Greer, & Torbeyns, 2006). The dual meaning of mathematical symbolism as *processes* to *do* and *concepts* to *know* has been captured in the term "procept" (Gray, Tall, & Pitta, 2000). Students can confuse fractions as *numbers* (that is, single quantities, as in $\frac{3}{4}$ of a litre) and fractions as *operations* (that is, proportions of quantities, as in $\frac{3}{4}$ of 10 metres). Physical materials can be used to model fraction tasks; different ways of modelling fractions include *region* or area models, *length* or measurement models, and *set* or collections models (van de Walle, 2004). Both region and length models involve *continuous* quantity, whereas set models involve *discrete* quantity. It has been suggested that one way to deal with all of this complexity is to define a fraction as simply "a point on a number line", allowing these different meanings of fractions to be deduced using logical reasoning (Wu, 2002). However, that has the disadvantage of excluding other powerful ways of coming to understand fractions using regions or sets.

Recent literature supports the idea that multiplicative thinking is essential for a deep and connected understanding of fractions, including proportions (Lamon, 2007; Thompson & Saldhana, 2003). It requires the recognition that "times" means "to envision something in a particular way – to think of copies (including parts of copies) of some amount" (Thompson & Saldhana, 2003, p. 104). It involves the realisation that there is an important reciprocal relationship, so that if quantity X is $\frac{1}{n}$ of a quantity Y, then Y is n times as large as X. Those who interpret fractions as "so many out of so many" (as in $\frac{1}{n}$ is "one out of n parts") are thinking of fractions *additively* instead of *multiplicatively* and will have difficulty dealing with situations where one quantity's size is a fraction of another quantity's size, when the quantities have nothing physically in common (for example, "the number of boys is what fraction of the number of girls?"). Hence, understanding 5×4 multiplicatively requires the understanding that the 4 in 5×4 is not just 4 ones (as in 20 = 4 + 9 + 7), but that the 4 is special because it is $\frac{1}{5}$ of the product (Thompson & Saldhana, 2003).

There is some debate in the literature about whether or not the teaching of algorithms is a good idea (Kamii & Dominick, 1998; Lappan & Bouck, 1998; Wu, 2002). Opponents of algorithms argue that some children never learn the algorithm and that those who can carry out algorithms don't always understand why or how they work, so they have little sense of when an algorithm is useful for solving a problem (Kamii & Dominick, 1998; Lappan & Bouck, 1998). They object to algorithms on the grounds that being told exactly how to do something "encourages children to give up their own thinking" (Wu, 1999, p. 4). Lappan and Bouck (1998) advocate the use of complex problems that encourage students to invent their own algorithms for adding and subtracting fractions. They argue that, although it takes more time to let students "wrestle with making sense of situations" than to show them an algorithm, it has the advantage of helping students learn to think and reason about mathematical situations (p. 184). The methods they subsequently develop can be efficient, powerful, and generalisable. Many western education systems now explicitly discourage teachers from introducing algorithms before children have developed a deep and connected understanding of part–whole relationships within the number system (for example, Ministry of Education, 2007a; National Council of Teachers of Mathematics, 2000).

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For some teachers, providing instruction on algorithmic procedures may be the only method they have available to convey information about fractions to their students because their own subject-matter knowledge in mathematics is not sufficiently strong. There is now a growing body of literature that recognises teachers' own knowledge of mathematics as making an important contribution to their effectiveness as teachers (Ball, Hill, & Bass, 2005; Ball, Lubienski, & Mewborn, 2001; Goya, 2006; Moch, 2004; Shulman, 1986; Zevenbergen, 2005). However, it is not enough simply to be a good mathematician. Teachers also need to understand ways to support the learning of their students in mathematics; that is, they need to have strong pedagogical content knowledge (see Ball, 2006; Shulman, 1986). In a recent study of teachers' knowledge of fractions, Ward, Thomas, and Tagg (this volume) found that although two-thirds of the 44 teachers they surveyed successfully identified that $\frac{3}{5} + \frac{2}{3}$ does not equal $\frac{5}{8}$ (that is, they did not make the "add across" error identified by Smith, 2002), only four of the teachers (9%) were able to describe clearly the "key understanding" that a student making the "add across" error needs to acquire in order to add unlike fractions successfully. Five of the remaining 40 teachers (11%) gave a response that showed some understanding of fractions. However, the other 80% gave either an incorrect response (66%) or no response whatsoever (14%). The fact that one-third of the teachers accepted the "add across" error as an appropriate strategy to use for adding fractions should also be of concern. Even teachers working with the youngest primary-school students need to have a strong understanding of fractions if they are to help their students move towards a deep and flexible understanding of the number system. It is important to note that the teachers in the study by Ward, Thomas, & Tagg had previously participated in the professional development programme offered as part of the Numeracy Development Projects (NDP), an initiative designed to enhance teachers' subject-matter knowledge of mathematics as well as to provide them with tools to support the mathematics learning of their students.

It has been suggested that teachers tend to perpetuate the ineffective practices of their own teachers by instructing their students in mathematics the way that they themselves were taught at school (Grootenboer, 2001; Zevenbergen, 2005). Given the challenges of getting to grips with the multiple meanings of fractions, it is not surprising that fractions is an area that many teachers can find difficult to teach. This makes them particularly vulnerable to adopting the algorithmic approach to teaching mathematics that was typical of their own mathematics teachers.

A systematic examination of the kinds of errors that students make with fractions has been done to help teachers detect and correct common mistakes made by students working with fractions (see Brown & Quinn, 2006). For example, Brown and Quinn found that, of the 27 year 10 students who were unable to find a common denominator when adding unlike fractions, more than two-thirds of them added the numerators and then added the denominators (that is, the "add across" error identified by Smith, 2002). A fifth of them showed misconceptions related to equivalent fractions. Only about half of their year 10 cohort of 143 students was able to add $\frac{5}{12} + \frac{3}{8}$ successfully.

Data from the NDP show that generally, students' knowledge of fractions is limited, with less than a third of students (approximately 30%) at the end of year 8 able to recognise the equivalence of the fractions $\frac{2}{3}$ and $\frac{6}{9}$ while ordering a collection of mixed fractions (Young-Loveridge, 2005, 2006, this volume). Similarly, only about a third of students at the end of year 8 were able to work out that, if $\frac{2}{3}$ of a particular number is 12, then that number must be 18 (Young-Loveridge, 2005, 2006, this volume).

The study reported here is part of a larger project that set out to explore the perspectives of year 7–8 students in six intermediate schools (see Young-Loveridge, Taylor, & Hāwera, 2005; Young-Loveridge, Taylor, Sharma, & Hāwera, 2006). As part of the interview, students were given some mathematics tasks, including one that involved addition of unlike fractions ($\frac{3}{4} + \frac{7}{8}$). The purpose of the analysis presented here was to examine students' responses to the adding fractions task.

Method

Participants

This study focuses on the responses of 238 year 7–8 students in six urban intermediate schools in the North Island. The sampling technique was designed to ensure that there were at least as many Māori and Pasifika students as European. Table 1 shows the composition of the sample by school decile, gender, ethnicity, year level, and mathematics ability (as assessed by their teachers). Approximately one-quarter (24%) of the sample were European, one-third (33%) were Māori, just over one-third (37%) were Pasifika, and a tiny group had both Māori and Pasifika ancestry (3%) or were Indo-Fijian (3%). There were slightly more boys (54%) than girls (46%) and slightly more students from year 8 (55%) than year 7 (45%). Three of the schools had participated in the NDP and three had not (non-NDP). The deciles of NDP schools ranged from 1 to 4 while those of non-NDP were between 3 and 6. Students came from a range of mathematics ability levels, and assessment data from schools was used to categorise students as low, medium, or high (low: PAT stanine 1–3 or level 3P and below on AsTTle; medium: PAT stanine 4–6 or level 3A to 4P on AsTTle; high: PAT stanine 7–9 or level 4A and above on AsTTle).

Table 1
Composition of the Sample by School Decile, Gender, Ethnicity, Year Level, and Mathematics Ability

School	**Gate***	**Hill***	**Ivy**	**Jute**	**Kite**	**Lake***	**Overall**
Decile	3	4	6	4	3	1	
Total	59	39	57	47	19	17	**238**
Gender							
Girls	26	20	24	22	10	7	**109**
Boys	33	19	33	25	9	10	**129**
Ethnicity							
European	15	17	12	13			**57**
Māori	21	11	33	14			**79**
Pasifika	22	8	9	13	19	17	**89**
Māori/Pasifika	1	3	3				**7**
Indo-Fijian				7			**7**
Year level							
Yr 7	28	21	20	23	7	9	**108**
Yr 8	31	18	37	24	12	8	**130**
Maths ability (as assessed by their teachers)							
High	7	15	7	7	8	0	**44**
Medium	37	13	30	26	7	8	**121**
Low	15	11	19	14	2	9	**70**
Unknown			1		2		**3**

*Schools that had participated in the NDP

Procedure

Schools were asked to nominate students from across a range of mathematics levels within each of the three main ethnic groups. Students were interviewed individually for about 30 minutes in a quiet place away from the classroom. They were told that the interviewer was interested in finding out their thoughts about learning mathematics. As well as questions about the students' views, a word problem involving the addition of $\frac{3}{4}$ and $\frac{7}{8}$ was given, as follows:

> Sione and Tama buy two pizzas. Sione eats $\frac{3}{4}$ of a pizza while Tama eats $\frac{7}{8}$. How much pizza do they eat altogether?

The task was read to students and they were offered a pencil to write down their problem-solving processes. They were then asked to explain to the interviewer their solution strategy, and these conversations were recorded on audiotape. Interviews were transcribed and the transcripts subjected to a content analysis to identify common themes coming through in the students' responses.

Results

Students' responses to the questions were organised according to common patterns emerging from the data. The code at the end of each excerpt indicates the school's name (initial letter) and the student's individual number, as well as year level and gender. Table 2 shows the number of students who responded in particular ways to the task.

Students' Strategies for Adding $\frac{3}{4}$ and $\frac{7}{8}$

Correct answer (13.4%)

A variety of answers were judged to be correct, including $1\frac{5}{8}$, $\frac{13}{8}$, $6\frac{1}{2}$ quarters, one and 2.5 quarters, and 1.6. (Note: Although 1.6 is only an approximation to the correct answer, it was accepted as correct. Likewise, $6\frac{1}{2}$ quarters, and one and 2.5 quarters were accepted, even though they violate the principle that the numerator and denominator should be whole numbers, because they are alternative expressions for the ratios $\frac{13}{8}$ and $1\frac{5}{8}$.) Only 32 students gave a "correct" answer to the problem. Several different approaches were taken to find the correct answer. Some students chose to use the "make a whole" strategy, similar to the "make ten" strategy, where part of one pizza was joined with the other pizza to make it into a whole pizza and the remaining fractional part calculated. Others chose to find a common denominator before adding the two fractional parts. This group was subdivided into two sub-groups: those who seemed to have a strong conceptual understanding of fractions and used appropriate fraction language to describe their strategy, and those who used language indicative of a procedural approach to solving the problem. A small group of students used quarters as the common denominator for adding the two fractional parts together.

Used the "make a whole" strategy and then calculated the leftover fractional part

Five students used the "make a whole" strategy successfully, partitioning one of the fractions so that part of it could be put with the other fraction to make it into a whole and the remaining fractional part calculated.

> Six plus seven because you take one off the six, which will make that [the six] five, plus that seven [from the $\frac{7}{8}$ pizza], which is eight over eight which is <u>one</u> [whole], and then the rest of it will be five over eight. (G33, yr 8 girl)
>
> [Drew two pizzas] Sione eats three-quarters, and I think that means eighths. Two of those eighths are left. One, two, three, four, five, six. Oh, here there's ... one, two, three, four, one and five-eighths [counting the remaining eighths in Tama's pizza]. (G46, yr 8 girl)
>
> Well, I took one of these and put them in that [$\frac{7}{8}$ pizza] and then that's <u>one</u> [whole], and then that would be five of them. (I14, yr 8 boy)
>
> [Drew two pizzas] I just shaded in seven-eighths and three-quarters and then I took one-quarter away from here [$\frac{7}{8}$ pizza], put it there [with the $\frac{7}{8}$ pizza] and it's <u>one</u>, and then I just added up five there, so I put five on. (I16, yr 8 girl)

Table 2
Number of Students Who Responded in Particular Ways to the Adding Fractions Task

School	**Gate***	**Hill***	**Ivy**	**Jute**	**Kite**	**Lake***	**Overall**	**%**
Total	59	39	57	47	19	17	**238**	
Correct answer								
Used "make a whole"	2		3				**5**	2.1
Used eighths as common denom. with understanding	1	4	1	1	2		**9**	3.8
Used eighths as common denominator with procedural explanation	1	2	3	1	1		**8**	3.4
Used quarters as common denominator			1	3	1		**5**	2.1
Other				2	1		**3**	1.3
No explanation	1			1			**8**	0.8
Total correct	*5*	*6*	*8*	*8*	*5*	*0*	*32*	*13.4*
% Correct	*8.5*	*15.4*	*14.0*	*17.0*	*26.3*	*0.0*	*13.4*	
Incorrect answer								
Used fraction equivalence	4	3	4	5	1		**17**	7.1
Ten-whole confusion	4			1		1	**6**	2.5
Estimated or guessed		2		1	2		**5**	2.1
Used "make a whole" strategy	1	1			1		**3**	1.3
Made a procedural error			1			1	**2**	0.8
Added nums/denominators ("add across" error)	13	7	12	15	4	3	**54**	22.7
Miscellaneous	9	5	11	6	1		**32**	13.4
Total incorrect	*31*	*18*	*28*	*28*	*9*	*5*	*119*	*50.0*
% incorrect	*52.5*	*46.2*	*49.1*	*59.6*	*47.4*	*29.4*	*50.0*	
No attempt	**23**	**15**	**21**	**11**	**5**	**12**	**87**	**36.6**
% no attempt	*39.0*	*38.5*	*36.8*	*23.4*	*26.3*	*70.6*	*36.6*	
Mathematics ability								
% high mathematics score	11.9	38.5	12.3	14.9	42.1	0.0	**18.4**	
% medium mathematics score	62.7	35.9	52.6	55.3	36.8	47.1	**45.2**	
% low mathematics score	25.4	28.2	33.3	29.8	10.5	52.9	**35.1**	

*Schools that had participated in the NDP

Used eighths as a common denominator and had strong conceptual understanding

Although a total of 17 students renamed $\frac{3}{4}$ as $\frac{6}{8}$ and used eighths as a common denominator for the addition of $\frac{6}{8} + \frac{7}{8}$, the explanations of the students differed markedly. Half of the students (n = 8) used language that suggested they had a strong conceptual understanding of adding fractions (see Figure 1 for the written explanation of one of the students in this group: K04). They referred frequently to the name of the fractional part as "eighths" and rarely or never used language such as "out of" or "over" when referring to the symbolic representation of fractions.

> I'm just working out how much that would be in eighths. So that would be thirteen-eighths. (H28, yr 8 girl)

> We go six-eighths and seven-eighths so they eat ... thirteen-eighths, then you could change that to one and five-eighths. (H29, yr 8 boy)

> I would make that into eighths to make it easier so it would be six-eighths, and I'd go six plus seven which is thirteen-eighths, which means a whole and five pieces. And then if there's eight pieces, I just minus the five off the second pizza because they've eaten an extra five of the second pizza, so you go eight minus five equals three. [Appeared to be working out how much pizza was <u>not</u> eaten, so was asked about how much <u>was</u> eaten] A whole and five pieces, eighths. (H30, yr 8 boy)

> I doubled the four to make it into an eight, and then I doubled the three to make it six-eighths, and then I added the two together. That gave me thirteen-eighths. And then I change it into a, I forget what it's called, this bit ... One and five-eighths. (K04, yr 8 boy)

> Thirteen-eighths. Because eight is two times four, I just doubled it so then that's six and that, that's eight, and then I added six to seven, which is 13 and then you put it over eight. [When asked about another way to work it out, gave the answer as 6.5 over 4] (G36, yr 8 girl)

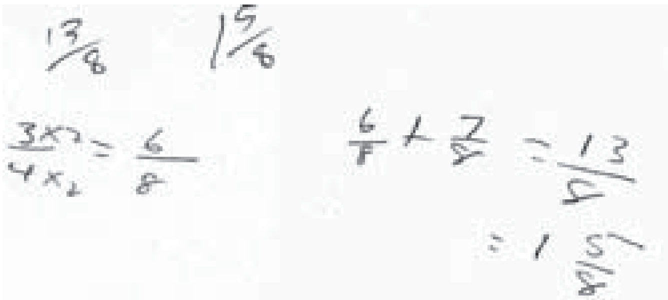

Figure 1: An example of the written explanation of a student who appeared to have strong conceptual understanding (K04)

Used eighths as a common denominator and gave a procedural explanation

Eight of the 17 students who renamed $\frac{3}{4}$ as $\frac{6}{8}$ so that they could add it to $\frac{7}{8}$ used language in their explanations that suggested the use of a highly procedural approach to solving the problem. For example, students in this sub-group described the use of "times by two" in converting quarters to eighths or "minus eight" from 13 to work out how many fractional pieces were left after the improper fraction was converted to a mixed number consisting of one whole and a fractional part. Virtually all of them used "over" to refer to the symbolic representation of fractions, as in "five over eight", rather than referring to the number and name of the fractional parts, as in "five-eighths". One student (H32; see Figure 2) multiplied the two denominators 4 and 8 to find a common denominator of 32 rather than using the relationship of eighths to quarters. One boy (K06) seemed to believe that the numerator must always be smaller than the denominator, as was evident from his comment after working out an answer of thirteen-eighths, that "you can't do that, so you ... minus it from eight." It is possible that some of these students did not fully understand the reason for using a common denominator or the acceptability of both an improper fraction and a mixed number as ways of expressing fractional number.

> I do times that by two to get eight, so I timesed that by two, and then I just added six and seven together which gave me 13, and then that was, I've forgotten what it was called ... [Interviewer: Improper?] Yeah, an improper fraction and then I just went 13 minus eight equals five, so I went one and five over eight. (H24, yr 8 girl)

> [On paper, wrote 32 as the denominator after multiplying four times eight, then wrote 24 for one numerator after multiplying eight times three, and 28 for the other numerator after multiplying four sevens, added them to get 52 in total, and wrote $\frac{52}{32}$] There is 20 left over, 20 over 32, and then I just broke it down to get five-eighths [he halved $\frac{20}{32}$ to get $\frac{10}{16}$, then halved again to get $\frac{5}{8}$], one and five-eighths. (H32, yr 8 boy)

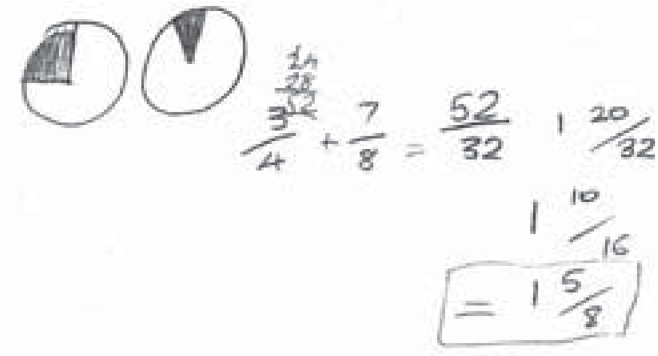

Figure 2: An example of a correct response showing the use of an algorithm (H32)

> I would do the numerals for both of them the same first. That's three over four. I would times it by two to get six over eight and then I would plus seven over eight plus six over eight which gives me 13 over eight and I would break it down to five over eight. It's one and five over eight. (J03, yr 8 girl)
>
> They eat ... Sione eats most of it except for one-eighth, oh no, he eats three-quarters, so there's one-quarter and one-eighth left. And altogether they would eat one whole pizza and five-eighths I think. Because there's one. I found out how much was left then I divided a pizza into eighths because I know what quarters are, and 'cause that's timesed by two. Like the bottom one, that's just double the amount to that so it's the double amount of pieces so you just cut each quarter into half. (I28, yr 8 girl)
>
> One and five-eighths. I changed three-quarters with six-eighths. And then I plus six-eighths with seven-eighths. And then I added it all together and it came out with fifteen-eighths, but then you can't do that. No, not 15. Thirteen-eighths and you can't do that so, you take out, you minus it from eight. And so, it becomes a five and put a one in front of the five so, it'll be five-eighths. One and five-eighths. (K06, yr 8 boy)

Used quarters as a common denominator

Five students chose to use quarters as the common denominator, changing seven-eighths into three and a half quarters in order to add it to three-quarters, and getting a total of six and a half quarters. Two of these students then worked out the answer as a mixed number, one and 2.5 (J16) or one and two and a half (J17). A sixth student, whose initial strategy involved using eighths, when asked for a different way of solving the problem, gave an answer of "6.5 over 4" (see G36 above). Four of the six students used language such as "out of" or "over" when describing their strategy.

> Six and a half quarters. I halved seven-eighths and that was three and a half quarters and then added three and a half to three. (J14, yr 7 boy)
>
> I know there's like half left over, sort of like six and a half quarters. I know that three, four, and then over here two-eighths equals a quarter so then I just do it like that. I just split it into halves. [Asked what the final answer was] Six and a half quarters. [Asked about other ways of saying it] Yes there is another way but I just need to think. Thirteen-eighths. I just timesed the three by two and then added it to the seven. (I21, yr 8 boy)
>
> They'd eat six point five. [Interviewer: six point five?] Yeah, six and a half [Interviewer: six and a half?] slices. [Interviewer asks for further explanation.] I halved it. I halved all this. [Interviewer: half of seven-eighths is?] It would be three point five over four. One pizza and two point five [Interviewer: two point five?] out of four. (J16, yr 8 boy)
>
> [Wrote 1 $2\frac{1}{2}$] You just halve those. Half of seven is three and a half over four, and then you added that to the three-quarters. And because six is more than the four, you need to, that means it's a whole plus ... one whole [pizza] and two and a half. (J17, yr 8 girl).
>
> There's a quarter pizza left on that one and on this one there's one-eighth left and then. Two, there'd probably be four-eighths left and then two out of eight and that's three-eighths. There'll be three out of eight left and three out of eight would ... [Asked to explain] He ate three and a half, six and a half, I think. I think it might be six and a half, I have no idea. I think it might be six and a half. (K01, yr 8 girl)

Other correct answers

There were also some idiosyncratic strategies used. For example, one student (J47) responded using decimal fractions, initially thinking there would be 1.25 pizzas but eventually settling on an estimate of 1.6 pizzas. One student (K02) started by making the "add across" error, adding the numerators 6 and 7, and putting the sum (13) over the sum of the denominators (16). However, during the course of her explanation, she self-corrected to give an answer of $\frac{13}{8}$, then converted this to a mixed number. Initially, she did not refer to the size of the pieces ("one full one and five pieces"), but in her written explanation it is clear that she was working with eighths (see Figure 3). Interestingly, her comment that "13 over eight doesn't work", before converting $\frac{13}{8}$ to a mixed number, suggests that she does not understand that improper fractions are acceptable ways of expressing fractional numbers.

> I think, like 1.25 pizzas, I think. Because the eight is the whole pizza so then there's, because that one is three-fourths and double that would be sort of expanding it so that's eight over six or six over eight and then six plus seven is 15, no, 13. And it would be 13 minus eight is five so then there's five over eight left. It's not 1.25, oh dammit, I forgot that would be 1.5 or 6, I think, like 1.6 pizzas, or something round there. [Asked to explain further] So it's an improper fraction. So there's more than one so you have to take away the eight from the 13. It's five. That's five-eighths of the pizza. (J47, yr 8 boy).

> Thirteen over 16, I think. I'm not too sure. [Asked to explain] I doubled this so it was the same as ... one pizza and five pieces. [Asked to explain further] I doubled that fraction, like I did here. And then I added the six and the seven, which gave me 13. And 13 over eight doesn't work, so I figured out the difference between 13 and eight and that gave me five, and then I took the five away from the 13, which gave me eight over eight, which is one full one and five pieces. (K02, yr 8 girl)

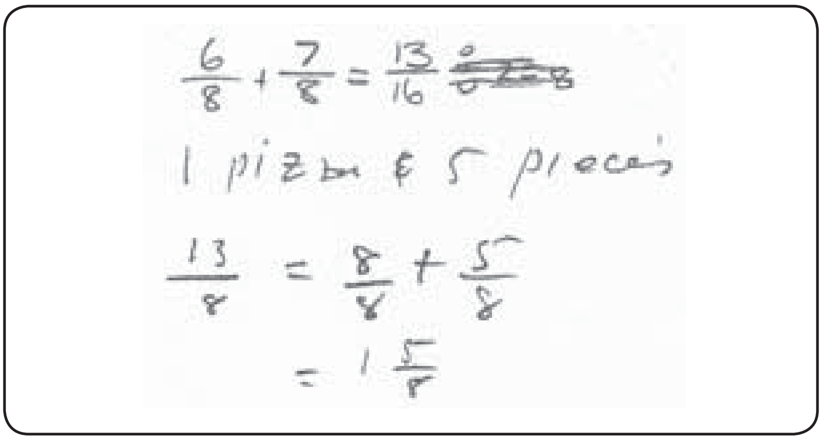

Figure 3: An example of a student who self-corrected her response during her explanation of her solution strategy (K02)

Correct answer without an explanation of the strategy

Two students were not able (or willing) to explain the strategy they had used to solve the problem. One boy (J26) claimed that he got his answer through making a lucky guess, but this response may have been because he was not completely confident he had solved the problem correctly.

> I don't know how. (G05, yr 8 girl)

> It's just a guess, one whole and five-eighths, I think. I'm not too good at fractions so I just thought of that. It was a lucky guess. (J26, yr 7 boy)

Incorrect answer (50.0%)

More than half of the students (n = 119) were unsuccessful in their attempt to add the two fractions. Several distinct strategies were evident from the students' responses. Some of the students who did not find a correct answer nevertheless showed an awareness of equivalent fractions. The majority of incorrect responders made the "add across" error, adding numerators and/ or denominators. A small group appeared to confuse the "make a whole" strategy with the "make ten" strategy (probably a familiar strategy used for adding whole numbers). Some students chose not to work out a precise answer, preferring instead to estimate (or guess) a close approximation to the answer. Three students attempted to use the "make a whole" strategy but miscalculated the remaining fractional part. Two made a procedural error while calculating the sum of the two fractions. Thirty-two students gave an idiosyncratic response that was unlike any other response given by someone in this group.

Used fraction equivalence to convert three-quarters to six-eighths

Seventeen students initially renamed $\frac{3}{4}$ as $\frac{6}{8}$, showing their awareness of fraction equivalence. However, they then went on to make an error, adding $\frac{6}{8}$ and $\frac{7}{8}$ together. Typically, they added the numerators and then added the denominators (the "add across" error) to get an answer of thirteen-sixteenths. One student (I25) began with an answer of "thirteen-eighths" but, after explaining his strategy, wrote "$\frac{13}{16}$". Some students ignored the denominator, adding just the two numerators (for example, H08). The language used by these students reflects a procedural approach with an emphasis on rules. For example, one student (G42) stated that "you can't have halves inside a fraction," whereas another (G44) stated that "you have to make the denominator the same."

> He ate three-quarters and that was six and he ate seven-eighths, so six plus seven is 13, and eight plus eight is 16 [wrote $\frac{13}{16}$]. (J34, yr 8 boy)

> Six and a half eighths, and then you have to change that again by doubling it so it's 16, that's thirteen-sixteenths. [Asked to explain further] Because you can't have halves inside a fraction. (G42, yr 8 boy)

> To me, it actually depends on whether there's four pieces on the first pizza or eight pieces on the second, so it would be, and that would be ten, it would be about, if that's eight pieces, then there must be eight pieces on that one as well. So that would be six and seven, and then I would add the seven and six together and get 13 out of 16. (H11, yr 8 girl)

> Thirteen-eighths, 'cause you know how to take in quarters. I cut them into eighths, this halves each part of it, and then double that number so I double that one. Doubled the six [? meaning doubled the three to get six]. Oh wait, I got that one wrong. That's an improper fraction isn't it? Oh the eight, thirteen-sixteenths. [Starts again] I cut each quarter in half ... I just plussed the two eights together and I plussed the six and seven to get 13 ... I was counting the two pizzas together, I just wanted to split them up ... I want to change that eight to 16 [wrote $\frac{13}{16}$] because I plussed these two together. (I25, yr 8 boy)

> They eat three-eighths altogether. [Asked to explain] Oh no, they <u>don't</u> eat three-eighths. They leave three-eighths, so they eat five-eighths altogether. [Asked to explain further] I doubled three-quarters to six-eighths. And I just saw that six-eighths, it's two-eighths away from a whole, and seven-eighths is, oh is seven over eight, is one over eight, yes, is a whole. So I added one over eight and two over eight which is three over eight, and then three over eight minus ... eight over eight is five over eight. [Asked to explain further] Three-quarters, so he eats this much and seven-eighths, half of that is 3.5 fourths, which would mean he'd eat that much and one of these so, that still there and that would be that one, so there'd be .5 of a pizza left, no um, one, no there'd be a quarter and a half which is three. There'd be eighths left, so there'd be three-eighths of one pizza left. [Asked how much pizza was eaten] Um, four, six, six and a half eighths, which is, yes, six and a half eighths which is thirteen-sixteenths. So they ate thirteen-sixteenths of a pizza ... So you can just double that so that's eight. Oh six-eighths and you just go six-eighths plus seven-eighths is thirteen-sixteenths. (G43, yr 8 boy)

> I just went, because you have to make the denominator the same, put four up to eight and then you just, because that's doubling it, so double three which is six-eighths and then add seven-eighths and six-eighths together, which is thirteen-sixteenths. (G44, yr 8 girl)

Ten–Whole confusion

Six students gave answers that revealed some confusion between the number of fractional parts making up the whole and the base-ten nature of the number system. This may have been the result of trying to use the "make a whole" strategy but confusing it with the "make ten" strategy they had previously used for adding whole numbers. For example, several students commented on particular number combinations that make ten (such as 7 and 3) instead of referring to the number of pieces needed to make a whole when the pieces are eighths. It may be for this reason that two other students (from the "miscellaneous" category) answered that adding $\frac{3}{4}$ and $\frac{7}{8}$ made "a whole one" (G51 and G57).

> A whole and two pieces. [? Treating the three-quarters as three-eighths, converting the $\frac{7}{8}$ pizza to a whole using one of the quarter pieces, leaving two quarter pieces left over?] [Asked to explain] Seven pieces plus three pieces equals one piece [? Thinking of a whole pizza as being like a "tidy ten", made up of combinations such as 7 and 3?] and then you get two more. [Asked to explain further] Three and four is seven, and seven and eight is 15 [? Adding the numerator and denominator for each of the two fractions?] (G29, yr 7 boy)

> I got one and three-quarters or one and a half. I'm not really sure. [Asked if he was making an estimate] Because what I got was two pizzas and a half, which can't be right. [Asked to explain how he got this] I just added those two – seven and three so that makes ... seven and three which makes a whole, and then I went eight plus four, which is 12, which is another whole and a bit. (G32, yr 7 boy)

Another student (H12) showed a similar confusion in reading a "teen" number as one whole and some fractional pieces (the number corresponding to the single-digit quantity beside the "1"), implying that 13 means one whole and three fractional parts.

> Ten [Asked ten what?] pieces. [Asked to explain] Oh, 13. [Asked to explain further] Because they just left eight pieces on a pizza. That's half there. [Interviewer says "and this is three-quarters."] That's the same as six eighths and seven eighths, so that equals 13. I plussed those two numbers. Oh, they ate one whole pizza and three-eighths. It's a mixed number. [Asked "What's a mixed number?"] One whole number and ... [Interviewer: So is this the <u>one</u> here, you're saying that <u>one</u> comes from the 13 there?] Yeah. So that's an improper fraction I think. [Interviewer: So 13 is equal to one and three-eighths?] Yes [had written $\frac{6}{8} + \frac{7}{8} = 13 = 1\frac{3}{8}$]. (H12, yr 7 girl)

Estimated or guessed the answer

Five students came up with answers that suggested they had tried to guess the answer rather than attempting to calculate a precise answer. This strategy may have been chosen because of uncertainty about an appropriate way of calculating the answer.

> Is that a third of the pizza left? That's three-quarters of a pizza and there's a quarter of it left, and so then there's another piece left that's about three-quarters, that's about a quarter. A third of it left. [Asked how much pizza they ate altogether] They had about a pizza and two-thirds. (I41, yr 7 girl)

> About a pizza and a quarter. Something like that. (I50, yr 7 boy).

> About a whole. [Was asked to explain] More than a whole. [Was asked to explain further] I forgot. [Interviewer offered him paper to write on.] Well, that there's three-quarters and that's seven-eighths so there'd be a little bit left in that one. It's about one and a half. (H33, yr 7 boy)

> One and seven-eighths. [Asked to explain] I don't know. Oh one and six-eighths. (K05, yr 8 girl)

> One and a quarter, I think. I don't really know. I just took away, I guess. (K09, yr 8 girl)

Tried to "make a whole" but miscalculated

Three students produced an incorrect answer as a result of calculation errors made while trying to use the "make a whole" strategy.

> I looked at three-quarters, and two-eighths equals a quarter, and take two away from eight [instead of seven] you get, six eighths. One whole and six-eighths. (K17, yr 7 boy)

> [Drew a diagram of two pizzas] That's one pizza and seven-eighths, that goes one ... one, two, three, four, five, six, seven, eight, and then shade seven. Seven and that one, shade. Shade that one in, and then that one. That one, if we just put that into there, that's one whole and a half. One whole pizza and half a pizza. (G09, yr 8 girl)

> One and a quarter. Because there's eight pieces in a pizza and Tama ate seven of them, and there's one more piece on that pizza, and Sione ate the last one from that eight and then two from the eight on the second pizza. The two they ate from the second pizza was a quarter. (H16, yr 8 boy)

Made a procedural error while trying to use an algorithm

Two students made errors while trying to use an algorithm, apparently not recognising the relationship between quarters and eighths. Lack of basic-facts knowledge (of 8 x 4) meant that the algorithmic procedure resulted in an incorrect answer (L02 thought 8 x 4 was 36; see Figure 4).

> [Wrote $(\frac{3}{4} \times \frac{8}{8}) + (\frac{7}{8} \times \frac{4}{4}) = \frac{24}{36} + \frac{28}{36} = \frac{52}{36}$. Wrote 52 – 36 = 16 using vertical written algorithm. Finally wrote $1\frac{16}{36}$] (L02, yr 8 boy)

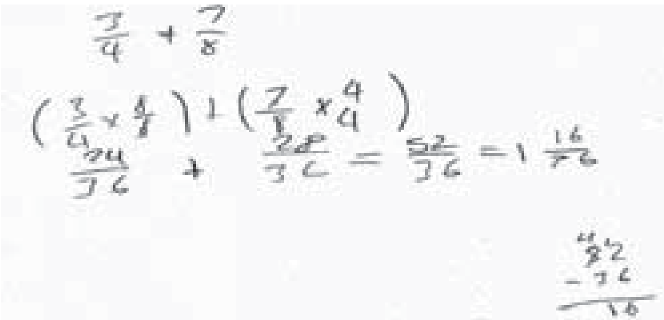

Figure 4: An example of an incorrect response showing the use of an algorithm (L02)

Added numerators and /or denominators

Adding the numerators and/or the denominators was the most frequent strategy used to get an incorrect answer, with at least 54 students using some form of this particular strategy (some of those who used fraction equivalence also went on to make this "add across" error). The most popular version of this strategy was to add three and seven for the numerator and four and eight for the denominator, giving an answer of $\frac{10}{12}$ (see Figure 5). Other variations on this strategy produced responses such as $\frac{10}{16}$, $\frac{12}{10}$, $\frac{10}{8}$, $\frac{20}{24}$, and 22 (the sum of all numerators and denominators). It was interesting to observe that five of the students who responded with $\frac{10}{12}$ then simplified it to $\frac{5}{6}$.

> [Wrote 3 + 7 = 10, 8 + 4 = 16, $\frac{10}{16}$] (G38, yr 7 girl)
>
> [Wrote 3 + 7 = 10, 4 + 8 = 12] The bottom number has to be bigger. (G40, yr 7 boy)
>
> [Wrote 3 + 7 = 10, 4 + 8 = 12, $\frac{10}{12} = \frac{5}{6}$] (H37, yr 8 boy)
>
> [Wrote $\frac{3+7}{4+8} = \frac{10}{12}$, $\frac{5}{6}$] (H40, yr 8 girl)
>
> [Drew two pizzas and shaded the eaten part. Wrote $\frac{3}{4} + \frac{7}{8} = \frac{10}{12}$] (J36, yr 7 girl)

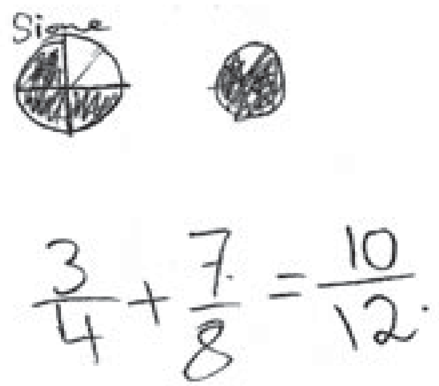

Figure 5: An example of a response showing the "add across" error (J36)

Miscellaneous responses

This category included unusual responses that were difficult to interpret, such as $\frac{1}{2}$, $\frac{45}{48}$, 2 pizzas, $\frac{1}{2}$ and $\frac{3}{4}$, 2 and a half, $\frac{6}{4}$ of 2 pizzas, $\frac{3}{4}$ and $\frac{1}{8}$, ten and a half, $\frac{9}{10}$, just over $\frac{3}{4}$, $\frac{3}{4}$ of 2 pizzas, and $\frac{52}{64}$ (the sum of cross multiplying with a denominator of eight times eight).

> Sione ate three-quarters and Tama ate seven-eighths, so that would be one whole pizza altogether. [Asked to explain] Well, seven-eighths is the same as three-quarters, well I think it is. So it would be one and a half. Yeah one pizza and a half. [Asked where the half came from] Like a decimal. I just did three times five, 15, and that would be one point five and that would be one and a half. (H05, yr 7 boy)

> I'm really bad at fractions. I've probably got it wrong. You have the pizza and you divide it into four because Sione ate three-quarters so she eats that part, that part and that part, and then you have Tama and he eats seven-eighths of the pizza so he eats that part and there's only that bit and that bit left. [Asked how much did they eat altogether] It's nearly two but not quite, I'm not sure 'cause I'm really bad at fractions ... I usually try really hard to understand what's going on but the fractions and stuff I don't get it and as much as I go over it and stuff, I just don't get it. [Interviewer comments on the usefulness of drawing pictures.] I'm pretty good with visualising things, like if sometimes we have problems, they'll give you like a set or something and they'll say what shape does it make and I can usually see in my head what shape it's going to make. (H31, yr 7 girl)

No attempt (36.6%)

More than a third of the sample (87 students) chose not to respond to the pizza problem. This group included three students who tried to draw the pizzas but then responded "Don't know" (see Figure 6).

Figure 6: An example of a drawing made by one of the students who eventually responded "Don't know" (J08)

As part of the ethics process, the students had been told that they could skip any question they did not want to answer. We respected that decision and did not press the students to make responses to the mathematics tasks. On reflection, we think it would have been useful to ask the students to draw something to show each of the two fractional quantities, even if they were not able to add them together. It is possible that this might have revealed some understanding of fractions by the students who chose to not even attempt the task. To find out which of the "no attempt" students might have been able to do the fractions task successfully if they had chosen to, we examined the relationship between students' responses to the fractions task and their assessed mathematics ability.

Relationship of Responses to Fractions Task with Assessed Mathematics Ability

Analysis of the mathematics assessment information showed that approximately ten (11.5%) of the students who made no attempt to do the task might have succeeded on the fractions task if they had tried to do it. These ten students had been assessed by their teachers as being at stage 7 or higher on the Number Framework, or in the upper third of the distribution (see Table 3). On the other hand, almost a fifth (19.2%) of the group who did not get a correct answer were from this high-mathematics-achievement group. It is interesting to note that a quarter (25.8%) of the students who found a correct answer were not among the high mathematics achievers.

It is clear from Table 3 that the majority of students who responded with a correct answer were among the highest mathematics achievers for their year level, with almost three-quarters (74.2%) of correct responders having been assessed as high in mathematics by their teachers.

Table 3
Numbers of Students Who Were High Mathematics Achievers for Each Response Type

Type and level of mathematics assessment	Correct response	**Response type** Incorrect response	No attempt
Number of students in group	n = 32	n = 119	n = 87
NDP Framework			
Stage 7	3	7	6
Stage 8	5	6	2
Total stage 7+	*8*	*13*	*8*
AsTTle tool			
Level 4A	2	3	1
Level 5B	1	1	
Level 5P	2	1	
Level 5A	1		
Level 6B		1	
Total level 4A+	*6*	*6*	*1*
Progressive Achievement Test			
Stanine 7	4	3	1
Stanine 8	3	1	
Stanine 9	2		
Total stanine 7+	*9*	*4*	*1*
Number of high maths achievers	*23*	*23*	*10*
Percentage of high maths achievers	*74.2%*	*19.2%*	*11.5%*

Discussion

Overall, relatively few students appeared to have a deep understanding of fractions or fraction computation. This finding is consistent with those writers who argue that fractions present a major challenge to students and to their teachers (Davis et al., 1993; Hunting, 1994; Lamon, 2007; van de Walle, 2004). However, this finding has some important implications for the implementation of New Zealand's new draft curriculum document, where the expectation is that students at level four should be able to solve problems using multiplicative and simple proportional strategies (Ministry of Education, 2006). Fewer than a third of year 8 students are at stage 7, advanced multiplicative, (see Young-Loveridge, 2005, 2006, this volume) and therefore few students in this study were able to add $\frac{3}{4}$ and $\frac{7}{8}$ fluently. The expectation that students at level four should be multiplicative thinkers is based on research evidence showing that students cannot engage with algebra effectively if they are not multiplicative thinkers (for example, Lamon, 2007; Wu, 2002). Hence, there is clearly a need to provide additional professional development for teachers working at the upper primary and intermediate levels (years 5–8) to help them to appreciate the importance of multiplicative thinking and provide them with instructional support in this area. The revisions to Book 1 (The Number Framework) and Book 6 (Teaching Multiplication and Division) are designed to do just that (see Ministry of Education, 2007b, 2007c). The Ministry of Education's fee-subsidy scheme, which provides some financial support to offset the costs of teachers doing further university study in mathematics education, may also help, but it needs far more publicity as well as support from schools if it is to have an appreciable impact on teachers' understanding of the upper stages of the Framework. Lamon (2007, p. 633) comments that "we educators are failing miserably at teaching the most elementary multiplicative concepts and operations."

In her longitudinal study using a design experiment, Lamon (2007) found that in the first two years of the programme, which was designed to build students' understanding of the central multiplicative structures involved in fractions, the students were outperformed on fraction computation by rote learners taught using a traditional approach that included rules and algorithms. However, in the longer term, the students whose instruction was focused on building meaning and sense making surpassed the rote learners. In the light of these findings, we need to exercise caution over how much improvement it is reasonable to expect from the NDP professional development programme in relation to the multiplicative and proportional domains of the Framework.

The findings of relatively limited fraction understanding by students in the present study raise some important questions about teachers' subject-matter knowledge in the domain of fractional number. Lamon (2007, p. 633) points out that many adults, including teachers, "struggle with the same concepts and hold the same primitive ideas and misconceptions as students do." This was borne out by a recent study of teacher knowledge about fractions (Ward, Thomas, & Tagg, this volume) where it was found that the majority of teachers in the study (91%), all of whom had participated in the NDP professional development programme several years prior to the study, were unable to articulate the key understanding needed to help students add fractions with unlike denominators. These two studies taken together underline the importance of strengthening teachers' subject-matter knowledge of fractions in particular. It is vital that this be made an urgent priority within pre-service teacher-education programmes as well as within in-service programmes.

A notable strategy used by a small group of students in the present study to add $\frac{3}{4}$ and $\frac{7}{8}$ was the "make a whole" strategy, whereby part of one fraction was put with the other fraction to make a whole and the remaining fractional parts counted (Huinker, 1998). This strategy is similar to the "make ten" strategy used in whole-number computation, where one of the addends is partitioned so that one of its parts can be joined with the other addend to make ten (or a multiple of ten), as in $9 + 5 = 9 + 1 + 4 = 10 + 4$ (see Thompson, 1999, 2000).

One group of students in the present study who found the correct answer did so using what appeared to be procedural knowledge rather than conceptual understanding. A crucial means of deciding whether or not the student's explanation was procedural was if their language suggested the application of taught procedures (Smith, 2002). Even though many of these students found the correct answer, their responses seemed very mechanistic and rule-based rather than being fluent and grounded in a deep and connected conceptual understanding of fractions. The absence of reference to the names of the fractional parts (for example, quarters and eighths), instead using positional language to describe the written symbols produced by the fraction computation (for example, "13 over eight", "five over eight"), was taken as an indication that they had used an algorithm involving the manipulation of the digits within the fractions according to a set of rules rather than carrying out meaningful computation with fractional quantities. This is consistent with Lamon's (2007) comment that research with students who have had at least five years of traditional instruction in mathematics shows that reasoning strategies tend to be replaced by rules and algorithms by the time students have been at school this long. Mack (1990) also found that her students referred to fractions in terms of the number of pieces rather than commenting on the size of the pieces.

A small group of students in the present study confused the "make ten" strategy with the "make a whole" strategy. These students seemed to have confused the decade-based (place value) structure of the number system with the particular kind of part–whole relationships found between the fractional parts and the whole when the whole is partitioned into parts other than ten (for example, eighths). Several students seemed to think that seven plus three made a whole, ignoring the denominators altogether. Another student thought that the number 13 meant that there was one whole and three fractional parts. This particular misconception suggests that a strong emphasis on the decade-based structure of the number system could be at the expense of other part–whole relationships that students need to understand. Teachers need to be aware that this potential misconception is one that some of their students may develop.

For a sizeable group of the students who gave an incorrect answer to the fractions tasks, it was clear from their explanations that they were aware of the need to find equivalent fractions when adding fractions with unlike denominators. Most of them knew that they needed to convert $\frac{3}{4}$ to $\frac{6}{8}$ in order to add it to $\frac{7}{8}$ (Huinker, 1998). A smaller group were aware of the equivalence of $\frac{1}{4}$ and $\frac{2}{8}$. The idea of fraction equivalence is a key component of the part–whole sub-construct for fractions (Behr et al., 1983).

About half of the students gave an incorrect answer to the fractions task, and half of these (about a quarter of the entire cohort) added the numerators and/or denominators, an error referred to as the "add across" error (Smith, 2002). The fact that the answer many students gave was less than one indicates that they were not thinking about the size of the individual fractions and the likely impact on the combination of the two fractions. With $\frac{7}{8}$ so close to one, and $\frac{3}{4}$ considerably more than $\frac{1}{2}$, it should have been obvious that the correct answer would be greater than one (Reys, Kim, & Bay, 1999). This finding points to the value of using benchmarks as reference points (e.g., 0, $\frac{1}{2}$, 1) as a way to help students appreciate the magnitude of particular fractions (Reys et al., 1999; van de Walle, 2004).

It was interesting to note that only a relatively small number of students drew diagrams to help them solve the pizza problem (26, of whom five gave a correct response, 18 gave an incorrect response, and three made no attempt). Only five of the students who got a correct answer used a diagram to help them, perhaps because many were able to solve the problem using abstraction. Although 18 of the students who produced an incorrect response drew diagrams, these were not always helpful. Three students drew diagrams but were unable to connect their intuitive understanding of fractions, as reflected in their diagram, with the formal written symbolism they had been taught at school. Like other writers, we believe that diagrams, as well as other physical materials, have much to offer in helping students to make sense of the problem by using pictorial representation (see Lamon, 2007, van de Walle, 2004). In our discussions with students about their perspectives on their mathematics learning, we got a clear impression that many of them viewed the use of physical materials as appropriate only for younger students or for students experiencing major difficulties with mathematics. It was interesting to note that the students who did use diagrams all used circular diagrams to depict fractions. Several writers (for example, Bay, 2001) have warned that the over-reliance on drawing pictures of pies may impede the development of a more abstract understanding of what a fraction is and thus slow down the acquisition of "the basic disposition towards algebra" (Wu, 2002, p. 60).

It is important to acknowledge that caution should be exercised in drawing firm conclusions about a student's conceptual understanding from just one task, as Mitchell and Clarke (2004) have pointed out. However, the use of a familiar context such as pizzas divided into quarters and eighths ought to have given the students the best possible chance to show any understanding of fractions that they did have. We recognise that there are other tasks that could provide insights about other aspects of students' understanding of fractional numbers. However, the advantages of being able to audiotape the entire conversation with each student in order to capture their verbal explanations as well as any written recording they may have done have helped to provide a rich source of data about students' thinking and problem-solving processes. The size of the cohort (n = 238), and the fact that it includes a substantial number of Māori and Pasifika students means that this data set may be able to provide some answers to many important questions, not just about students' mathematical thinking but also how that is related to their perspectives and views about their mathematics learning at school and beyond.

The findings of this study underline the importance of working towards ensuring that students having a deep and connected understanding of fractions, beginning this process from the earliest years at primary school. It is important for teachers to recognise the importance of their own subject-matter knowledge of mathematics and its impact on the learning of their students and to take responsibility for addressing their own learning needs in this domain. The strategies identified in this study provide a useful starting point for teachers in relation to pedagogical content knowledge. For example, the "make a whole" strategy could provide a useful alternative to other more conventional methods of solving a problem involving addition of fractions. This could be part of an approach that capitalises on familiar elements of the Number Framework currently used to develop students' understanding of whole numbers (see Mack, 1990; van de Walle, 2004). For example, students could be shown how to *count* with fractions, beginning with the easiest and most familiar fraction, $\frac{1}{2}$, and using cardboard semi-circles to model the counting process, as in $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, and so on. This might be following by counting verbally without materials (that is, *imaging*). This experience with counting might help students to appreciate that they can use the same processes to count units that are fractional parts as they already use to count units of one or units that are multiples, such as fives or tens. Progressing to other fractions such as $\frac{1}{4}$ would allow fraction equivalence to be experienced within the context of counting, as in $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $1\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$, 2, and so on. This could be followed by adding fractions, initially with sums within a whole, but later with sums beyond a whole, using the "make a whole" strategy as a way of using knowledge of partitioning to break the addition process into steps; first making one of the fractions into a whole by joining it with part of the other fraction, then adding the remaining fractional part. *Compensation* strategies could be modelled, as in $\frac{3}{4} + \frac{7}{8}$ is the same as $\frac{3}{4} + (1 - \frac{1}{8})$, so $1\frac{3}{4} = 1\frac{6}{8}$ and $1\frac{6}{8} - \frac{1}{8} = 1\frac{5}{8}$. *Doubling* and *halving* strategies could also be explored, as in $1\frac{1}{2} + 1\frac{1}{2} = 3$, or half of $1\frac{1}{2}$ is $\frac{1}{2} + \frac{1}{4}$, which is the same as $\frac{3}{4}$. A similar approach could be used with subtraction, starting with counting back by fractional units, then using "bridging through a whole" (similar to "bridging through ten"; see Thompson, 1999, 2000) to subtract across wholes, as in $1\frac{1}{4} - \frac{5}{8}$ is the same as $1\frac{2}{8} - (\frac{2}{8} + \frac{3}{8}) = 1 - \frac{3}{8}$ and $\frac{8}{8} - \frac{3}{8} = \frac{5}{8}$. By drawing on some of the key processes that are used in building whole number understanding, teachers would be helped to appreciate some commonalities between the whole-number system and the rational-number system. This kind of approach might also help students to develop flexibility in unitising and reunitising quantities (Lamon, 2007). (Note: Lamon uses the term *unitising* (p. 630) to refer to "the process of mentally chunking or restructuring a given quantity into familiar or manageable conveniently-sized pieces in order to operate with that quantity"). The ideas suggested above would strengthen the measurement sub-construct for fractions, the sub-construct Lamon believes provides one of the best starting points for building understanding of rational numbers.

It is important to acknowledge that *time* is a key issue in coming to understand fractions. According to Lamon (2007), "multiplicative ideas, in particular, fractions, ratios, and proportions, are difficult and develop over time" (p. 651). This is supported by her research findings, that in the longer term, it was the deep and connected understanding of fractions acquired by students in the experimental classrooms that provided them with the power and flexibility to perform meaningful operations and eventually to surpass the rote learners. Lamon's research findings underline the importance for educators of being patient but persistent in bringing about change in the teaching of fractions. It also supports the call for longer and more sustained professional development for teachers working with middle-years students, who need to become multiplicative thinkers if they are to engage productively with algebra at secondary school.

The challenge for us now and in the future is to ensure that students do not give up the search for sense making and understanding in mathematics or turn to procedural approaches for solving problems (van de Walle, 2004). However, this is no mean feat. It requires a major shift for teachers in ways of thinking about the goals of mathematics learning. The emphasis must be on building conceptual understanding at all levels of the school. Fractions provide an ideal context in which to take on this challenge.

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