What Do Teachers Know About Fractions?

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This paper reports on the development and initial trialling of a tool to assess teacher knowledge of the teaching of fractions. Results showed that a pen-and-paper assessment focusing on teachers' pedagogical content knowledge can be both efficient and effective in differentiating between teachers. In general, teachers scored more highly on questions requiring a response based on their content knowledge than on questions where responses required them to describe the key ideas involved or the actions they would take with a student. Questions that caused teachers difficulty involved the addition of fractions, division with fractions, and proportional reasoning, with approximately one-third of teachers' responses indicating a lack of conceptual understanding in each of these areas. These results suggest that the knowledge of teachers may be a factor limiting student achievement, particularly in the proportions and ratios domain of the Number Framework. Further work is required to establish a link between teachers' scores in the assessment and student achievement data. If the validity of the assessment tool can be established in this way, its use to tailor teachers' professional learning may be significant.

Background

It is widely accepted that the knowledge a teacher holds affects the way they perform all the core tasks of teaching (National Council of Teachers of Mathematics, 2000). In particular, a teacher's knowledge of subject matter, student learning, and development and their teaching methods have all been identified as important elements of teacher effectiveness (Hammond & Ball, 1997). Focusing on a teacher's knowledge of content, Shulman (1986) defined pedagogical content knowledge (PCK) as knowledge of a subject "for teaching". He differentiated this from pure subject knowledge by describing PCK as including the best ways to present the subject to learners, the most useful examples to use to illustrate certain points, and an idea of the misconceptions and preconceptions that learners may bring with them to the learning.

Since Shulman's definition of PCK, researchers have worked in many subject areas to investigate teacher knowledge and map the precise knowledge a teacher requires to be effective (Hill, Schilling, & Ball, 2004). In mathematics, these investigations have included comparing the views of pre-service teachers with those of experienced teachers (Ball, 1990), in-depth interviews with practising teachers (Harel & Lim, 2004), and comparisons of the differences in teacher knowledge across cultures (Ma, 1999). In focusing on teacher knowledge in mathematics, researchers have made a distinction between teachers that have an algorithmic or rule-based understanding of mathematics and those that have a deep conceptual understanding.

Difficulty with fractions among teachers is well documented in many countries, and many authors consider fractions to be the most difficult area of mathematics covered in primary school (Smith, 2002). Studies into teacher knowledge of fractions have found both procedural and conceptual understandings among teachers, although procedural understandings dominate in this area (Fuller, 1997). Considerable differences have also been found in the explanations teachers provide to students when working with fractions. Some teachers use significant conceptual information in their explanations, while some focus more on algorithms and procedures (Leinhardt & Smith, 1985). When looking at the representations that teachers use to present fractions to students, a limited repertoire has been found (Ball, 1990). Circular representations are most commonly used, but these can be problematic because they are unable to illustrate conceptually-complex operations with fractions, such as division.

In addition to defining the knowledge required to teach mathematics effectively, recent studies have also sought to measure teacher PCK in mathematics (Hill, Schilling, & Ball, 2004). This work has been followed closely, with investigations of the relationship between student gains and teacher knowledge. These investigations have revealed a non-linear relationship between student achievement data and teacher PCK scores. In particular, the teachers in the bottom third of the knowledge distribution have a significant negative impact on their students' achievement (Hill, Rowan, & Ball, 2005). For those working in professional development, this work highlights the need to differentiate between teachers on the basis of their existing knowledge and to provide focused professional development for the third of teachers with the least knowledge.

In New Zealand, the Numeracy Development Projects (NDP) have dominated professional development initiatives in mathematics education over recent years. The NDP aim to improve students' use of mental strategies to solve number problems by focusing on the professional knowledge of teachers, and there is much evidence to suggest the projects have been effective in raising student achievement (Bobis, Clarke, Clarke, Thomas, Wright, Young-Loveridge, & Gould, 2005). Despite this increased achievement, teacher knowledge remains an area of concern. A recent report from the Education Review Office claims that 23% of teachers hold only partially effective or not effective pedagogical content knowledge in mathematics (Education Review Office, 2006). Alongside this, student achievement data from the NDP indicate that multiplicative and proportional thinking remain areas of difficulty with student performance "a little disappointing" in these domains (Young-Loveridge, 2006).

> The data indicate that there are important issues to investigate further with respect to multiplicative thinking and understanding of fractional numbers. (Young-Loveridge, 2006, p. 20)

This paper reports on work undertaken in 2006 to develop an efficient and effective tool to assess teacher knowledge of the teaching of fractions. This information could be used to tailor professional development programmes to meet the needs of individual teachers.

Method

Development of the Assessment Tool

The assessment tool was developed with the assistance of a reference group consisting of five primary teachers and a numeracy facilitator experienced in working with teachers in the NDP. The group met three times with the researcher to discuss the PCK required by teachers to be effective in teaching fractions. This group provided feedback on the form of the assessment, assisted with the drafting of questions, conducted a small-scale trial of alternative wordings for questions, and helped modify the assessment on the basis of these trial results.

The final version of the assessment tool comprised seven questions based on scenarios involving the teaching and learning of fractions. In general, questions described a scenario and then asked teachers to first identify the mathematically correct answer to the problem posed and then describe either the key understandings involved or the teacher actions required. Each question required two or three responses, depending on the nature of the content covered, with a total of 17 responses required to complete the assessment. All questions included an option for teachers to indicate if they were unsure of how to respond.

The content areas covered in the assessment were the comparison and ordering of fractions, addition of fractions, proportions, fractions greater than one, division of fractions, and equivalent fractions.

Trial of the Assessment Tool

The assessment tool was trialled in two schools selected from the 22 primary schools involved in the 2006 Longitudinal Study. The Longitudinal Study has been ongoing since 2002 and examines the impact of the NDP in schools that have been involved for a number of years (Thomas & Tagg, 2004, 2005, 2006; Thomas, Tagg, & Ward, 2003). It uses a sample of schools selected to be representative of the national sample.

One high-decile school and one low-decile school were selected to be involved in the trial of the assessment tool. Both schools were located in the North Island, and each had approximately 20 classroom teachers. Table 1 displays this information. Participating teachers were predominantly full-time classroom teachers, but part-time classroom teachers and management staff were also included.

Table 1
Participating Schools

	School one	School two
Decile	1	8
First participation in NDP	2001	2002
Number of participating teachers	21	23

One of the senior management team at each school administered the assessment, following a set of instructions provided by the researcher. The assessment was administered at staff meetings, with teachers completing it individually. Teachers were given as much time as required to complete the assessment.

A feedback questionnaire was used to gather information on participating teachers' perceptions of the assessment. The questions asked teachers to describe their feelings about the assessment and to rate and describe their own knowledge of fractions.

Analysis of Results

Teacher responses to each of the questions were grouped and used to develop marking criteria. In general, three categories of answer were identified. These were: correct answers that identified the key concepts involved and explained or described these fully; answers that showed some understanding by identifying key concepts but omitted an explanation or description of these; and incorrect answers that either identified irrelevant or unrelated concepts or were too general and broad.

The marking criteria developed were used to mark the assessments completed by the teachers, with each correct response scoring one point and a total of 17 points possible. An analysis of scores and responses was then undertaken. Where reported percentages do not add to 100, this is due to rounding error.

Findings

Usefulness and Practicality

In general, the assessment was practical to administer and produced a range of scores that could be used to differentiate teachers. The average time taken to complete the assessment was approximately 14 minutes, with the times for completion ranging between 5 and 22 minutes. Table 2 shows the spread of teacher scores. One teacher recorded correct responses for all questions and received full marks of 17 points; one teacher received a score of zero, with no correct answers. Almost every score possible was recorded by at least one teacher.

Table 2
Teacher Scores

Score	0	2	3	4	5	6	7	8	9	10	11	12	13	14	17
Number of teachers	1	1	1	2	6	5	2	7	2	3	3	5	4	1	1

Responses in the feedback questionnaire showed that most of the teachers involved did not find the assessment stressful. Thirty-six percent of teachers were neutral or relaxed about being asked to do the assessment, while another 36% reported feeling slightly anxious. Ninety-three percent of the teachers reported either being more relaxed after the assessment or no change in their level of anxiety before and after the test, and just 5% of the teachers felt more anxious after the assessment. In general, teacher comments reflected an open approach to completing the assessment.

> I think I've done ok!
>
> I know more than I thought I did.
>
> Glad I could do it (I hope I got it right.)

Teacher Knowledge of Fractions

Table 3 shows participating teachers' scores in each of the content areas covered. As described previously, the first part of each question explored the teachers' own knowledge of fractions (noted in the table by shaded rows) and the later parts probed teachers' conceptual understanding and their ability to communicate this understanding to students – their PCK. Questions had either two or three parts, depending on the nature of the scenario.

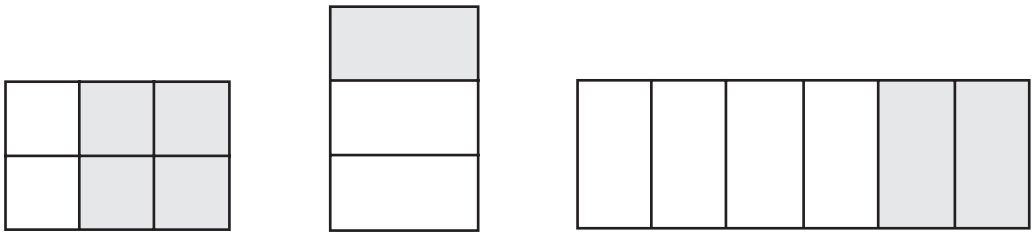
Table 3
Teacher Scores by Question Content

Shaded rows indicate content-only questions; non-shaded rows indicate PCK questions.

	Responses			
	Correct % (n)	Show some understanding % (n)	Incorrect % (n)	No response % (n)
1. Comparison and ordering of fractions	59 (26)	na	36 (16)	5 (2)
	55 (24)	na	36 (16)	9 (4)
	32 (14)	9 (4)	11 (5)	48 (21)
2. Addition of fractions	66 (29)	na	14 (6)	20 (9)
	9 (4)	11 (5)	66 (29)	14 (6)
3. Proportional reasoning	68 (30)	na	27 (12)	5 (2)
	43 (19)	9 (4)	11 (5)	36 (16)
	50 (22)	9 (4)	11 (5)	30 (13)
4. Improper fractions	84 (37)	na	2 (1)	14 (6)
	45 (20)	34 (15)	7 (3)	14 (6)
5. Division involving fractions	48 (21)	na	39 (17)	14 (6)
	59 (26)	5 (2)	14 (6)	23 (10)
	18 (8)	9 (4)	36 (16)	36 (16)
6. Equivalent fractions	89 (39)	na	9 (4)	2 (1)
	18 (8)	39 (17)	27 (12)	16 (7)
7. Ordering unit fractions	86 (38)	na	2 (1)	11 (5)
	48 (21)	20 (9)	20 (9)	11 (5)

In general, teachers scored more highly on questions requiring a response based on their content knowledge than on questions where responses required a description of the key ideas involved or the actions they would take with a student. On questions involving content, the average percentage of teachers recording correct responses was 69%, while on questions involving PCK, the average percentage of teachers recording correct responses was 36%. This trend is illustrated by teacher responses to question 6, which assessed teacher knowledge of equivalent fractions (see Figure 1).

Which shape has $\frac{2}{3}$ of its area shaded?



Mark insists that none of the shapes have $\frac{2}{3}$ of their area shaded.

Do any of the shapes have $\frac{2}{3}$ of their area shaded?
What action, if any, do you take?

Figure 1: Question 6, equivalent fractions

Eighty-nine percent of teachers correctly identified that one of the shapes had two-thirds of its area shaded. However, only 18% of teachers clearly described how to use materials to demonstrate that two-thirds is equal to four-sixths. One of these responses is illustrated in Figure 2.

Show equivalence. $\frac{4}{6} = \frac{2}{3}$. Rearrange visually to show $\frac{2}{3}$ over the top of $\frac{4}{6}$.

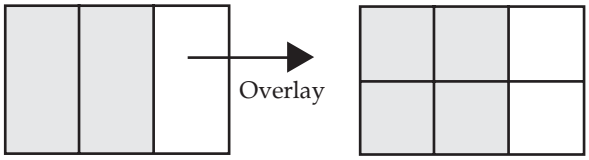


Figure 2: Teacher response giving clear description of appropriate teaching action

Thirty-nine percent of teachers were able to give a response that showed some understanding by mentioning equivalent fractions and/or the use of materials but did not clearly describe the actions they would take. Examples include:

> Discuss and use materials to learn about equivalent fractions.
>
> Equivalent fractions need to be modelled and taught.
>
> Develop notion of equivalence.

The question for which fewest teachers answered both parts correctly involved the addition of fractions (see Figure 3).

> You observe the following equation in Sally's work: $\frac{3}{5} + \frac{2}{3} = \frac{5}{8}$
>
> Is Sally correct?
>
> You question Sally about her understanding and she explains: I scored 3 goals out of 5 in the first half and 2 goals out of 3 in the second half. Overall, I scored 5 out of 8 goals.
>
> What, if any, is the key understanding she needs to develop to solve this problem?

Figure 3: Question 2, addition of fractions

Sixty-six percent of teachers successfully identified the equation in Sally's work as incorrect. However, just 9% (four teachers) clearly described the key understanding she needed to develop in order to solve the problem. These responses included:

> That the denominator is the total number of goals or parts for each fraction. That the denominator is the total attempts and the numerator is the number of times she succeeded. $\frac{3}{8} + \frac{2}{8} = \frac{5}{8}$
>
> That she is working with 8 as the whole, therefore it was $\frac{3}{8} + \frac{2}{8} = \frac{5}{8}$

In response to the second part of the question, 11% of teachers recorded an answer that showed some understanding, while 80% of teachers were unable to identify the key understanding required, recording either an incorrect response (66%) or no response at all (14%). This was the only question for which the majority of the teachers recorded incorrect responses. Of the 29 responses categorised as incorrect, 12 teachers recorded answers that were very general and 17 teachers described the use of a common denominator to solve the problem.

> She needs to understand that you need to have a common denominator to add fractions together.
>
> She needs to change the fractions so she has the same denominator.

One teacher's response illustrated a lack of understanding of common denominators:

> Sally needs to understand about using a common denominator before adding fractions:
>
> $\frac{3}{5} + \frac{2}{3} = \frac{9}{15} + \frac{10}{15} = \frac{19}{30}$?

The use of a common denominator is a valid mathematical approach when adding three-fifths and two-thirds, but in this situation, the key understanding that Sally needs to develop is not related to the use of a common denominator. If a common denominator is used in this example, the numbers in the equation ($\frac{9}{15} + \frac{10}{15} = \frac{19}{15}$) bear no real meaning to the goals scored in the game. Sally's understanding of how to use fractions to successfully represent the goals in the two halves of the game will not be developed by developing her knowledge of common denominators. These responses indicate a rule-based or procedural approach to tasks involving the addition of fractions rather than a deep understanding of the concepts involved. The NDP seek to promote a deep understanding among teachers where:

> less emphasis is given to the rote performance of written algorithms to calculate answers. (Ministry of Education, 2006, p. 2)

A general lack of understanding of the content involved in the addition of fractions was indicated by 34% of teachers, who either stated that Sally's equation was correct (14%) or omitted to answer (20%) the first part of the question. Addition and subtraction with fractions is placed at stage 8 of the additive domain in the Number Framework. Student achievement data indicate that there are at least a small number of students operating at stages 7 and 8 from year 3 on (Young-Loveridge, 2006). Of the 15 teachers that recorded an incorrect response to this question, eight teachers identified that they teach students from these year levels. This finding may be a cause for concern because it suggests an area in which teacher knowledge may be impacting on student achievement.

Division with fractions

Another question on which teachers performed poorly involved division with fractions (see Figure 4).

> You observe the following equation in Jane's work: $1\frac{1}{2} \div \frac{1}{2} = \frac{3}{4}$
>
> Is she correct?
>
> What is the possible reasoning behind her answer?
>
> What, if any, is the key understanding she needs to develop to solve this problem?

Figure 4: Question 5, division involving fractions

Forty-eight percent of the teachers correctly identified that there was a problem with the equation in Jane's work, while 39% of teachers believed her recording was correct. Just 18% were able to clearly describe the key understanding that Jane needed to solve the problem by clarifying the conceptual question behind the equation. These responses included:

> It's asking "how many halves are there in $1\frac{1}{2}$?"
>
> That the question is asking "if $1\frac{1}{2}$ is a half, what is the whole?"

Seventy-two percent of teachers were unable to identify the key understanding that Jane required. This group included 36% who recorded no response to the question and 36% who gave an incorrect answer. Incorrect answers were very general or identified unrelated concepts. Examples include:

> Equal parts – sharing, mixed fractions, equivalent fractions
>
> $\frac{1}{2} + \frac{1}{2} = \frac{2}{2}$, denominators don't change.

Fifty-nine percent of teachers identified a plausible explanation as Jane's possible reasoning. The explanations were of two types. One was the addition of the numerators and denominators to give three over four, and the other was one and a half being divided by two instead of by a half.

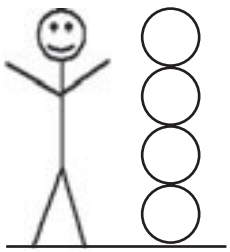
> She may have added 1 + 1 + 1 = 3 then added 2 + 2 = 4.
>
> She is dividing one and a half into 2 groups.
>
> She's just halved one and a half.

It is interesting to note that five of the 12 teachers who identified Jane's reasoning as one and a half divided in two evenly believed that her answer to the problem was correct. In general, responses to this question indicate that many of the teachers who participated lacked a conceptual framework for problems involving the division of fractions. This is consistent with other work in this area, where it has been found that, in general, teachers have a limited conception of fractions (Ball, 1990), with many unable to apply their general understanding of division into the context of fractions. Commonly, teachers regard division with fractions problems primarily as fractions problems rather than division problems. The use of language has also been found to be problematic in this area: to divide something in half is to divide it into two equal parts; to divide something by one-half is to form groups of a half. This is a distinction that teachers find difficult (Fuller, 1997).

Proportional reasoning

Disappointing student performance in the multiplicative and proportional domains over recent years has been identified as an area requiring further investigation (Young-Loveridge, 2006). For this reason, it is interesting to consider teacher responses to proportional reasoning problems such as question 3 (see Figure 5).

This is Mr Short.



The height of Mr Short is 4 large buttons.
The height of Mr Tall is 6 large buttons.

When paper clips are used to measure Mr Short and Mr Tall, the height of Mr Short is 6 paper clips.

What is the height of Mr Tall in paper clips?

Jim answers 8 paper clips.
Steve answers 9 paper clips.

Who is correct?

What is the possible reasoning behind **each** of their answers?

Figure 5: Question 3, proportional reasoning
An adapted version of the Mr Tall/Mr Short problem (Khoury, 2002)

Sixty-eight percent of teachers were able to identify Steve's response of nine paper clips as the correct answer to this problem. Of the remaining 14 teachers, 10 teachers mistakenly identified Jim's answer as correct, two teachers believed both Jim and Steve's answers could be considered correct, and two teachers did not record an answer. In total, nearly 32% of teachers were unable to answer this question correctly.

When asked to describe the students' thinking, 43% of teachers were able to describe Jim's additive reasoning clearly:

> Jim has simply retained the difference of 2 between each height.
>
> Buttons + 2 = paper clips
>
> Jim thinks there will be 2 paper clips difference, as with the buttons.

Fifty percent of teachers were able to describe Steve's proportional reasoning by describing the proportional gain or the difference in proportion between the two heights:

> Steve looked at the percentage by which each increased, Mr Short = 4 x 150% = 6, therefore Mr Tall = 6 x 150% = 9.
>
> Steve has recognised an increase of half and subsequently added half of 6, making it 9.
>
> Ratios 2:3, 4:6, 6:9

The inability of approximately one-third of the participant teachers to correctly answer this proportional reasoning question is potentially a cause for concern. Student achievement data suggests that teachers of students in years 5 and above will have at least a small number of students who are proportional thinkers in their classrooms (Young-Loveridge, 2006). Of the 14 teachers that were unable to answer this question, five identified that they were teaching in classes with students that were year 5 or higher. These results provide evidence that the knowledge of teachers may be a factor limiting student achievement in some cases and are worthy of further investigation.

Teacher Scores by Year Levels

In the development and trial of the assessment tool, many teachers working in year 1–3 classes expressed a belief that they did not require a comprehensive knowledge of fractions because this was not an area their students worked in extensively.

> Know enough to teach junior end of the school. Would have to do a refresher course for myself if I had to teach level 3.
>
> I teach juniors and we really only cover $\frac{1}{2}$, $\frac{1}{4}$ at this stage.

The results did provide some evidence of a lower level of knowledge among teachers working in junior classrooms. The 13 teachers working solely with students in years 0–2 received an average score of 7.8 in the assessment, while the 25 teachers working with students in year 3 and above received an average score of 9.9. Six teachers did not identify the level of the students they taught or identified that the students they taught had special needs. In addition, 46% of the junior teachers identified themselves as being unsure of how to respond to at least one question, while just 16% of the remaining teachers identified themselves as being unsure of one or more responses.

While it is difficult to define the precise level of knowledge required by teachers in junior classrooms, there is some justification for teachers at this level not needing to reason proportionately because they are unlikely to be required to teach proportional reasoning. Student achievement data indicate that by year 2, approximately 4% of students are able to find a fraction of a number, order unit fractions, and use symbols for improper fractions (Young-Loveridge, 2006). While there are relatively few year 2 students working with fractions, teachers need to have a sound conceptual understanding of fractions, and operations with fractions, in order to meet the learning needs of these students.

Teachers' Perceptions of Their Own Knowledge

Participating teachers were asked to rate their own knowledge for teaching fractions using a five-point scale ranging from very weak to very strong. In general, those teachers that rated their own knowledge as weak achieved lower scores on the assessment than those who rated their own knowledge more highly. Table 4 shows these results.

Table 4
Teacher Self-assessment and Average Scores

Self-assessment	Teachers % (n)	Average score
Very weak	7 (3)	4.7
Weak	20 (9)	7.2
Moderate	15 (22)	9.4
Strong	18 (8)	10.3
Very strong	2 (1)	11.0

While these results provide some evidence that self-assessment may be of value in assessing the knowledge of teachers, the large range of scores achieved by teachers within each rating limits the value of this information. For example, within the 22 teachers who rated themselves as having a moderate knowledge for teaching fractions, scores in the assessment ranged evenly from 3 to 17 points.

Concluding Comment

Results suggest that a pen-and-paper assessment focused on teachers' PCK can be both efficient and effective in differentiating between teachers. In general, teachers scored more highly on questions requiring a response based on their content knowledge than on questions where responses required a description of the key ideas involved or the actions they would take with a student.

Questions on which teachers performed poorly involved the addition of fractions, division with fractions, and proportional reasoning. Teacher responses in these areas indicated a lack of conceptual understanding, with 39% of teachers describing the use of common denominators to add fractions in a way that did not support conceptual understandings and 36% not being able to clarify the conceptual question behind an equation involving division with fractions. Thirty-two percent of teachers were unable to answer proportional reasoning problems correctly, providing some evidence that the knowledge of teachers may be a factor limiting student achievement in some cases.

Teachers working with students in year 3 and above scored more highly than teachers working with students in year 1 and 2, and this is consistent with participants' views that teachers of younger students do not need a highly developed knowledge of fractions. However, it needs to be noted that students from year 2 on will be working with fractions and teachers at this level need to have a sound conceptual understanding of fractions, and operations with fractions, in order to meet the learning needs of these students. Teachers' perceptions of their own knowledge in fractions tended to reflect their scores in the assessment; however, the large range of scores achieved by teachers within each self-assessment rating limits the value of this information.

Further work in this area is required to establish a link between teachers' scores in the assessment and student achievement data. If it can be shown that teachers with poor scores in the assessment have correspondingly poor student achievement data, the validity of the tool will be established and its use to tailor teachers' professional learning may be significant. It may also be useful to develop similar assessments based on other areas of numeracy teaching and learning.

References

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal*, 90, 449–466.

Ball, D. L. and Bass, H. (2003). Making mathematics reasonable in school. In G. Martin (Ed.), *A research companion to Principles and Standards for School Mathematics* (pp. 27–44). Reston, VA: National Council of Teachers of Mathematics.

Ball, D. L., Hill. H. C., & Schilling, S. G. (September, 2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal, 105* (1), 11.

Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, R., Young-Loveridge, J., and Gould, P. (2005). Supporting teachers in the development of young children's mathematical thinking: Three large scale cases. *Mathematics Education Research Journal, 16* (3), 25–57.

Education Review Office (2006). *The quality of teaching in years 4 and 8: Mathematics.* Retrieved 21 July 2006: ww.ero.govt.nz/ero/publishing.nsf/Content/Reports%20-%20National%20Reports

Fuller, R.A. (1997). Elementary teachers' pedagogical content knowledge of mathematics. *Mid-Western Educational Researcher, 10* (2), 9–16.

Hammond, L. D. & Ball, D. L. (June, 1997). *Teaching for high standards: What policymakers need to know and be able to do.* Retrieved 21 June 2006: http://govinfo.library.unt.edu/negp/Reports/highstds.htm

Harel, G. & Lim, K.H. (2004). Mathematics teachers' knowledge base: Preliminary results. *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education,* 28th, Bergen, Norway, July 14–18, 2004.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effect of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42* (2), 371–407.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 105* (1), 11–30.

Khoury, H. A. (2002). Exploring proportional reasoning: Mr. Tall/Mr. Short. *Making sense of fractions, ratios and proportions: 2002 yearbook.* Reston, VA: NCTM.

Leinhardt, G. & Smith D. A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology*, 77, 247–271.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* New Jersey: Lawrence Erlbaum.

Ministry of Education (2006). *Book 3: Getting Started.* Wellington: Ministry of Education.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*, Reston, VA: NCTM.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15* (2), 4–14.

Smith, J. P. III (2002). The development of students' knowledge of fractions and ratios. *Making sense of fractions, ratios and proportions: 2002 yearbook.*

Thomas, G. & Tagg, A. (2004). *An evaluation of the Early Numeracy Project 2003:* Exploring issues in mathematics education. Wellington: Ministry of Education.

Thomas, G. & Tagg, A. (2005). Evidence for expectations: Findings from the Numeracy Development Project longitudinal study. *Findings from the New Zealand Numeracy Development Project 2004* (pp. 21–34). Wellington: Ministry of Education.

Thomas, G. & Tagg, A. (2006). Numeracy Development Project longitudinal study: Patterns of achievement. *Findings from the New Zealand Numeracy Development Projects 2005* (pp. 22–33). Wellington: Learning Media.

Thomas, G., Tagg, A., & Ward, J. (2003). *An evaluation of the Early Numeracy Project 2002:* Exploring issues in mathematics education. Wellington: Ministry of Education.

Young-Loveridge, J. (2006). Patterns of performance and progress on the Numeracy Development Project: Looking back from 2005. *Findings from the New Zealand Numeracy Development Projects 2005* (pp. 6–21). Wellington: Learning Media.