Base 2 and Complementary Arithmetic

AC
EA
AA
AM
AP

We are solving problems in number contexts
We are exploring binary mathematical contexts
We are exploring the properties of numbers

Exercise 1: Making Up

And what makes 9?

1)	2	(2)	3	(3)	7
4)	1	(5)	5	(6)	0
7)	6	(8)	4	(9)	8

And what makes 99?

10)	20	(11)	50	(12)	10
13)	12	(14)	84	(15)	67
16)	33	(17)	54	(18)	23

And what makes 999?

19)	222	(20)	444	(21)	888
22)	123	(23)	321	(24)	246
25)	454	(26)	015	(27)	007

And what makes 9999?

28)	1234	(29)	6789	(30)	5555
31)	2468	(32)	3579	(33)	1717
34)	0018	(35)	0020	(36)	3142

And what makes 99999?

37)	12345	(38)	56789	(39)	11111
40)	02468	(41)	03030	(42)	97631
43)	00200	(44)	50	(45)	351

What do you notice about making numbers up to 9?

I notice…

Exercise 2: Make a Complement

The complement of a number “xxx” is <u>and what makes</u> “999?”

The complement of 123 is 876. Note 1 + 8 =9 and 2 + 7 = 9 and 3 + 6 = 9.
The complement of 46 is 53. Note 4 + 5 = 9 and 6 + 3 = 9.

Work out the complements of these numbers

1)	234	(2)	222	(3)	252
4)	1234	(5)	5555	(6)	0000
7)	18	(8)	45	(9)	99

What do you notice about finding complements?

I notice

Work out the complements of these numbers

10)	222	(11)	44	(12)	8
13)	101	(14)	32123	(15)	0006
16)	45	(17)	9999999	(18)	121212121

Do you notice anything more?

I notice

The number 246 is made up of 200 + 40 + 6. This can be written in expanded form as 2 x 100 + 4 x 10 + 6 x 1, and is part of learning about place value. Writing numbers like this should be something you have already learned and practised.

Write these numbers showing the place values in expanded form.

19)	34	(20)	789	(21)	5555

Write these numbers showing the place values in expanded form. The different characters are being used for different digits (so think of ☺✡ as a two digit number, were each digit is different)

22)	☺✡	(23)	★✦✪	(24)	✪✪✪✪

What is the complement of each of these numbers?

25)	34	(26)	☺✡	(27)	789
28)	★✦✪	(29)	5555	(30)	✪✪✪✪

Exercise 3: Complementary Subtraction

This is a cool trick. You are going to learn how to subtract two numbers by adding.

The problem 57 – 21 is easily seen to have the answer 36. There is no "carry" or "renaming" to be done so the tens digits are subtracted and the unit digits are subtracted. In the tens place 50 – 20 = 30 and in the units or ones place 7 – 1 = 6.

The problem 51 – 27 requires a bit more thinking and understanding of how numbers work. Fifty-one (51) can be renamed forty-eleven (40 + 11). This is a good choice for this problem but is not the only choice available. The answer is now 40 – 20 = 20 in the tens place and 11 – 7 = 4 in the ones or units place. The answer is 20 + 7 = 27.

BUT… like most ideas in mathematics there is another way.

1) In the first problem 57 – 21 we calculate the complement of 21.

The complement of 21 is 78.

Now add 57 and 78. We add 50 + 70 to get 120 and then 7 + 8 to get 15.
The answer is 120 + 15 = 135

What do you notice about 135 and the answer to 57 – 21?

I notice

2) In the second problem 51 – 27 we do the same procedure. It is just as easy.

The complement of 27 is 72. We add 51 + 72 and get 120 + 3 = 123.

What do you notice about 123 and the answer to 51 – 27?

I notice

You might have noticed that the answer is always 100 bigger and 1 smaller than the real answer. Now for a really good question!

3. Why is it that the complement answer is always 100 bigger and 1 smaller than the actual answer to a 2-digit subtraction problem?

I think it is because…

Exercise 4: More Complementary Subtraction

Complementary subtraction works with many digit numbers. In this exercise we use base 10 numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. In a later exercise we use base 2 numbers 0, 1 and see it is very simple to do. Computers use complementary base 2 addition to subtract.

First let's practice our complementary addition in base 10.
Here is a 3 digit subtraction problem solved using complementary arithmetic.

521 – 257 = 521 + 742 -1000 + 1 = 12 x 100 + 6 x 10 + 3 x 1 - 10 x 100 + 1 x 1 = 264

Your turn. Complete these 3 digit subtractions using complementary arithmetic.

1) 631 – 243

2) 876 – 387

3) 315 – 076

Make a comment about complementary subtraction. Do you notice anything?

Humm…

Now try and solve these problems in the same way. Make sure you add 0 place holders when you need. That is why the 0 is in the hundreds place in (3) above.

4) 83 – 57

5) 8765 – 1875

6) 227 – 78

7) 2709 – 883

8) 66666 – 777

Can you think of a reason why humans use base ten? Is it the only base we have ever used? Martians have one finger on each hand and use base 2. Venusians have 3 fingers on one hand and 4 on the other. What base would they use?

Answer here **Earthling**

Exercise 5: Martian Maths $1_2 + 1_2 = 10_2$

A very very old mathematician's joke reads "There are 3 types of mathematicians; those that can count and those that can't." After you have done this activity you will understand a similar joke that says "There are 10 types of mathematicians; those that can count and those that can't."

Counting can be done in a very large number of ways. The simplest is based on the number two. Martians count this way. All of the Martians I have met had one finger on each hand and one toe on each foot. They are very good at pointing.
A Martian only uses 0's and 1's when counting. The following table shows the start of counting using Martian numbering. See if you can finish the pattern. Look carefully.

Human	Martian		Human	Martian
Zero	00000		14	
1	00001		15	
2	00010		16	
3	00011		17	10001
4	00100		18	
5	00101		19	
6	00110		20	
7			21	
8	01000		22	
9			23	
10			24	
11			25	
12	01100			
13				11011

Explain the Martian counting system in your own words.

My words!

1) Invent a way to make sure that numbers written in Martian (like 101) do not get confused with numbers written in Earthling (like 101). Discuss your convention with your teacher.
2) Explain the joke about 10 types of mathematician
3) If each finger on your hand is a new column of a Martian number, how many can you count to, Earthling, using the Martian System?
   (a) on one hand? (b) on two hands?

(c) what is the largest number you can count to using your body bits?

Exercise 6: Complements in Base 2

This is the easiest of all exercises. You could be a Martian and get these right.

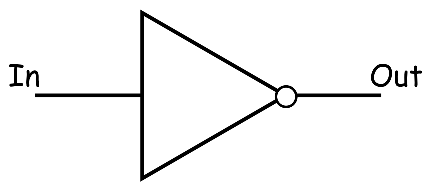
The complement of 0 is 1 and the complement of 1 is 0 in base 2. All we do to find the complement of a number in base 2 is to flip all the digits between 0 and 1.

The complement of 00101011 is 11010100. You are more likely to loose track of where you are up to than get the "flipping thing" wrong.

Try these problems.

1)	101	(2)	111	(3)	1010
4)	0110	(5)	01010101	(6)	11111111
7)	01100110	(8)	11000011	(9)	00000001

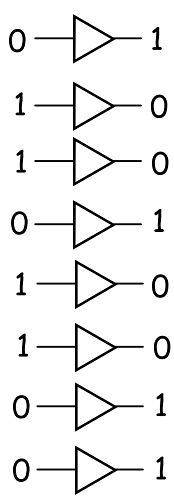
This is so easy that a little electronic circuit was made up to do this job. The circuit is called an inverter. It looks like this. The little o means the output is opposite of the input. These circuits come in sixes on the 4049 chip.



Logic Chart	
In	Out
0	1
1	0

In electronic circuits a "0" means "off" or no voltage
and a "1" means "on" or 9 Volts in many circuits.

To "flip" or invert an 8 bit binary number like 01101100 we use 8 inverters all working in parallel or "processed at the same time". It looks like this.



Experience Task

Make sure the power is OFF.

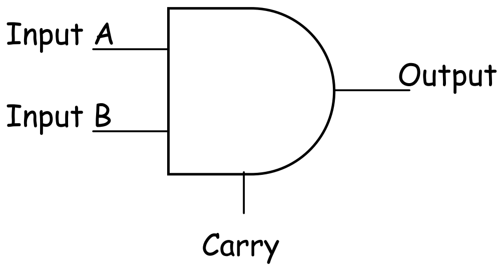
Find an old computer and open it up. Look for the highways of 8 and 16 tracks running around the circuit board. These look like highways joining little townships of computer chips.

These "highways" are the parallel tracks that convey all the information in and around the main processor. The highways you are looking at probably connect hard disks or CD drives to the main circuit board.

Look for a hex-inverter chip in Dick Smith Electronics.

Exercise 7: Computers, Adding

Another little electronic device computers use is called an AND gate. The AND gate is represented by the symbol as shown and behaves with this logic.



Logic Chart			
Input A	Input B	Carry	Output
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

This all looks quite complicated. A little explanation and all will be well.

In Base 2 arithmetic the basic addition facts are

0 + 0 = 00
0 + 1 = 01
1 + 0 = 01
and 1 + 1 = 10.

Look at the Logic Chart and note how the adder circuit mimics this arithmetic. Making 8 adder chips work in parallel (see Exercise 6) makes a circuit that can work with numbers up to 11111111_2 in size. If you did Exercise 5 you will be able to see that this is the number 255 in base 10 which is not very big at all. But it is a start. Modern computers use 64 or 128 bit adding. How big are these base two numbers?

Try adding these 4-bit numbers in base 2

1) 1110 + 0001 (2) 1100 + 0010 (3) 1001 + 0110

Add the corresponding parts just as you do in the decimal system.
When you have 1 + 1 you need to "carry" just as you do in the decimal system.

Try adding these 4-bit numbers with one carry in each.

4) 1110 + 0010 (5) 1100 + 0100 (6) 1001 + 0001

Here are some mega problems with lots of carries.

7) 1100110011 + 0011111111 (8) 00001111 + 00001111 (9) 1111 + 1111

Note: Many of these problems show what happens to a computer when you run out of room …"overload error".

Task: Make up some of your own base 2 addition problems and work out the answers.

Exercise 8: Computers, Multiplying

Look at the following base 10 multiplication problems.

4 x 6 = 6 + 6 + 6 + 6 3 x 20 = 20 + 20 + 20

Write a statement about what you see.

If you can see that multiplying is just a better and more efficient way of adding then you are exactly correct. Multiplying can be thought of as "repeated addition".

This is exactly how computers multiply.

4 x 6 becomes 6 (once) 12 (twice) 18 (thrice) 24 (four-times)….stop, display the answer. Inside is a little counting circuit that counts the number of times and stops when it gets to 4.

Here is an algorithm or programme that will multiply two numbers.

Start
10 Get first number (4)
20 Store in memory Count
30 Get second number (6)
40 Store in memory A
50 Store in memory B
60 Add memory A and memory B and store in memory C
70 Move C to A
80 Decrease Count by 1
90 Is count equal to 1 (NO…Goto 60, YES Goto 100)
100 Display C
End

Check Table			
A	B	C	Count
6	6	12	4
12	6		3

TASK A

The Check Table lets you think through the logic in steps just as a computer would do very quickly. Your task is to finish the chart. Start from the beginning.

TASK B

Make a new check table and solve these problems.

1) 3 x 7 (2) 8 x 9 (3) $101_2 \times 11_2$

Exercise 9: Computers, Dividing

Dividing two numbers is one of the slowest operations that a computers does. This is because it is a multi-step procedure and takes longer.

How do you think a computer would divide 12 by 3.

My answer

Here is a programme or algorithm that a computer could use to do 12 divided by 3.

Start
10 Get first number Numerator
20 Store in A
30 Get second number Denominator
40 Store in B
50 Set Count to 0
60 Find complement of B
70 Store in C
80 Add A to C and store in D
90 Take off 10 and add 1 to D and store in A
100 Increase Count by 1
110 Test if A = 0 (NO… Goto 80, YES…Goto 120)
120 Display Count as the answer.
End

Check Logic Table					
A	B	C	D	Count	
12	3	6	18	0	
9		6	15	1	NO
6		6	12	2	NO
3		6	9	3	NO
0				4	YES

TASK A
Work through the programme and see if you can repeat Check Logic Table correctly.

TASK B
Make a new check table and solve these problems.
1) 20 divided by 4 (2) 75 ÷ 15 (3) $1000_2 \div 10_2$

Exercise 10: Computers, Powers and other Functions

Computers can only do the operation of addition and they only do this using binary numbers. They can also find the complement of a number.

How does a computer do these mathematical operations?	
Addition	Uses addition ability of computer
Subtraction	Uses complementary addition
Multiplication	Uses repeated addition
Division	Uses repeated subtraction and counts
Powers	Uses repeated addition repeatedly

The computer does all of this in binary and at about half of the speed of the main processing chip oscillator. This is more than 200Mhz or 100 million calculations per second. Modern computers use several processors running together and are at least 10 times faster again.

TASK

Choose one of the following topics and investigate more about it on the internet and in your school library. Make up a colourful A3 poster recording what you find out. Include pictures, diagrams and questions. Your teacher will be able to use this for wall displays in the classroom and you may be asked to present you project to other students.

LIST (select one)

History of computers
Binary arithmetic
Cost of memory chips – 'yesterday and today'
Types of computers
Computers in Spacecraft
The Shuttle Computers Systems
Writing a programme to sort three numbers from smallest to largest
Write a program to show how a computer would work out 2^3. Create a check logic table to show that it works properly
Other Topic…check with teacher.

Computers Can Only Add
Answers

Exercise 1

1)	7	(2)	6	(3)	2
4)	8	(5)	4	(6)	9
7)	3	(8)	5	(9)	1
10)	79	(11)	49	(12)	89
13)	87	(14)	15	(15)	32
16)	66	(17)	45	(18)	76

19)	777	(20)	555	(21)	111
22)	876	(23)	678	(24)	753
25)	545	(26)	984	(27)	992

28)	8765	(29)	3210	(30)	4444
31)	7531	(32)	6420	(33)	8282
34)	81	(35)	9979	(36)	6857

37)	87654	(38)	43210	(39)	88888
40)	97531	(41)	96969	(42)	02368
43)	99799	(44)	49	(45)	648

I notice…

There are patterns in the digits of some numbers - like 18 become 81. Overall, each digit can be considered on its own without looking at the other digits.

Exercise 2

1)	765	(2)	777	(3)	747
4)	8765	(5)	4444	(6)	9999
7)	81	(8)	54	(9)	00

10)	777	(11)	55	(12)	1
13)	898	(14)	67876	(15)	9993
16)	54	(17)	0000000	(18)	878787878

I notice…

There is not much extra to notice than was the case for exercise 1

19) 3 × 10 + 4 × 1 (20) 7 × 100 + 8 × 10 +9 × 1

21) 5 × 1000 + 5 × 100 + 5 × 10 + 5 × 1

22) ☺ × 10 + ✡ × 1 (23) ★ × 100 + ✦ × 10 + ✪× 1

24) ✪ × 1000 + ✪ × 100 + ✪ × 10 + ✪ × 1

25) 65 (26) (9 - ☺) × 10 + (9 - ✡) × 1 (27) 210

For these next questions, square brackets have been invented to show what is happening in each column of the number, and to keep each column apart.

28) [9 - ★][9 - ✦][9 - ✪] (29) 4444

30) [9 - ✪][9 - ✪][9 - ✪][9 - ✪]

Exercise 3

1) You should notice the number is 99 more than the answer (100 bigger and 1 smaller).

2) You should notice the number is 99 more than the answer (100 bigger and 1 smaller).

3) The reason is not obvious. In the 1st problem 57 -21 we create the following chain. 57 - 21 becomes 57 + 99 – 21 = 57 + 100 – 21 - 1 = 100 + 57 - 21 - 1 = 100 bigger and 1 smaller than the original problem as suggested in the exercise.

Exercise 4

1) $6 \times 100 + 3 \times 10 + 1 \times 1 + 7 \times 100 + 5 \times 10 + 6 \times 1 - 10 \times 100 + 1 = 388$
2) $8 \times 100 + 7 \times 10 + 6 \times 1 + 6 \times 100 + 1 \times 10 + 2 \times 1 - 10 \times 100 + 1 = 489$
3) $3 \times 100 + 1 \times 10 + 5 \times 1 + 9 \times 100 + 2 \times 10 + 3 \times 1 - 10 \times 100 + 1 = 239$

I notice…
Comment will vary, though and important one is that the system only works when subtracting numbers that have the same columns, so for 347 – 56, the complementary addition is 943, not just 43.
Author notes the method is quite easy to do but a bit tedious.

4) $8 \times 10 + 3 \times 1 + 4 \times 10 + 2 \times 1 - 10 \times 10 + 1 = 26$
5) $8 \times 10000 + 7 \times 1000 + 6 \times 10 + 5 \times 1 + 8 \times 10000 + 1 \times 1000 + 2 \times 10 + 4 \times 1 - 10 \times 10000 + 1 = 6889 + 1 = 6890$
6) $2 \times 100 + 2 \times 10 + 7 \times 1 + 9 \times 100 + 2 \times 10 + 1 \times 1 - 10 \times 100 + 1 = 149$
7) $2 \times 1000 + 7 \times 100 + 0 \times 10 + 9 \times 1 + 9 \times 1000 + 1 \times 100 + 1 \times 10 + 6 \times 1 - 10 \times 1000 + 1 = 1819 + 6 + 1 = 1826$
8) $6 \times 10000 + 6 \times 1000 + 6 \times 100 + 6 \times 10 + 6 \times 1 + 9 \times 10000 + 9 \times 1000 + 2 \times 100 + 2 \times 10 + 2 \times 1 - 10 \times 10000 + 1 = 65889$

Probably base 3.5 at a rough guess.

Exercise 5

The pattern is obvious to Martians, but Earthlings may not have much of a clue. You could try a web search on the binary system, or discuss the place value system for Martian mathematics with your teacher

1) The convention used by mathematicians is to put a subscript after the number, so 101_2 shows the number is in base 2, while 101_7 means the answer is in base 7. We don't put a number for base ten, as this is the 'normal' base we usually use, so we only have to show when we are NOT using base ten.
2) 2 in Martian is written as 10
3) (a) One hand 31 (one less than 2^5) = 11111_2
   (b) Two hands $2^{10} - 1 = 1023 = 1111111111_2$
   (c) Two hands and two feet and your head is $2^{21} - 1 = 2097154 = 111111111111111111111_2$

Exercise 6

1)	010	(2)	000	(3)	0101	(4)	1001	(5)	10101010
6)	00000000	(7)	10011001	(8)	00111100	(9)	11111110		

Exercise 7

64 bit gives $2^{64} - 1$ as the biggest number. Try doubling to work out this number – or have a look at it on your calculator, or on a computer.
128 bit gives 1^{128} - 1

1)	1111	(2)	1110	(3)	1111
4)	10000	(5)	10000	(6)	1010
7)	10000110010	(8)	00011110	(9)	11110

Exercise 8

Multiplications can be written as repeated subtractions

Task A

Check Table			
A	B	C	Count
6	6	12	4
12	6	18	3
18	6	24	2
24		24	1

Task B

1)

Check Table			
A	B	C	Count
3	3	6	7
6	3	9	6
9	3	12	5
12	3	15	4
15	3	18	3
18	3	21	2
21			1

(2)

Check Table			
A	B	C	Count
8	8	16	9
16	8	24	8
24	8	32	7
32	8	40	6
40	8	48	5
48	8	56	4
56	8	64	3
64	8	72	2
72			1

(3)

Check Table			
A	B	C	Count
101	101	1010	11
1010	101	1111	10
1111			01

Exercise 9

The computer would repeatedly subtract, so for 12 ÷ 3 it would repeatedly subtract three and see how many times this happened – if the computer could subtract! Instead it will have to repeatedly add the complement of three until it gets to the number 12 and then say how many times it had to do this.

A	B	C	D	count
20	4	5	25	0
16		5	21	1
12		5	17	2
8		5	13	3
4		5	9	4
0				5

A	B	C	D	count
75	15	84	159	0
60			144	1
45			129	2
30			114	3
15			99	4
0				5

A	B	C	D	count
1000	0010	1101	10101	0
0110		1101	10011	1
0100		1101	10001	10
0010		1101	1111	11
0				100

Exercise 10

Project and wall display