Newsletter 90: November 2010

Introduction

Despite the fact I've been around mathematicians for over 50 years they still manage to amaze me with the work they do. I had several ideas for this 90th newsletter but I decided to Google '90' to see what came up. It told me all sorts of interesting (?) things, like 90 being a nontotient. I had no idea what that was nor what a Perrin number was so I kept fossicking through the web to see what all these unknown names were about. And now I'm amazed at what people have found interesting with numbers and presumably have spent much of their lives trying to track down and understand. It's a pity I didn't have more time to find out if any of these esoteric looking ideas had been applied successfully anywhere. But let me tell you some of the things that I followed up.

Take the Perrin numbers for example. They start off like this: 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39. Can you see the pattern there? There's nothing you can do with 0, 2 and 3, but after that you add two of the earlier numbers to get the next number. So 0 + 2 = 2 (the fourth number); 2 + 3 = 5 (the fifth number); 3 + 2 = 5 (the sixth number); and you can probably see how you can get everything else from there.

This is fascinating but there are three things that worry me. First, wasn't Perrin just stealing Fibonacci's idea? Second, why did he start off with 0, 2, 3 rather than 1, 2, 3, say? And third, what possible use could that all have been?

Basically Fibonacci started off with 1 and 1 and added them to get 2; then he added 1 and 2 to get 3 and so on. So this basic idea of adding two consecutive numbers to get a third has been known since 1215 when Fibonacci wrote his book to introduce the new decimal system to Europe. His sequence of numbers had no other use than to demonstrate how much easier arithmetic was in base 10 than in the Roman system. It's always a surprise to me that the I, II, III, IV, V, VI, VII, VIII, IX, X method lasted as long as it did. You really needed to be a genius to use it to divide CCXXXIV by XVIII. Maybe you had to be a genius to just be able to work out what numbers they really were.

Anyway, if we know about Fibonacci's sequence we now manage to see it sun flowers and somehow the limit of two consecutive Fibonacci numbers appears as the golden ratio, that number supposedly beloved of Ancient Greek architects.

So maybe Perrin thought that the copyright was up on Fibonacci's work and it was time to stir the old idea up again. But one of his thoughts was that the *n*th Perrin number wasn't divisible by *n*. If this was true, it might lead to a quick test to see if a number was a prime or not. This has been the Holy Grail of number theorists for some time. You may think that you can find out if a number is prime or not by just rattling through the possible factors, but have a crack at 1223334444555556667666555554444333221. Oh alright it's not prime, but I'm sure that I could find an odd 337-digit number that wouldn't be so easy to factor – even with a rather big computer.

Now it turns out that this dream of number theorists has a, temporarily at least, application. And that is the RSA system of encoding (see http://en.wikipedia.org/wiki/RSA, for example). In the middle of all that there is another seemingly crazy and complicated idea and that is of finding the number of numbers less than a given number, *n*, that have no factors in common with that number *n*.

For example, if n = 6, 1, and 5 are less than 6 and have no factors in common with it and there are 2 of them.

Now if a number is equal to this number of numbers less than a number that have no factors in common with it, it's called a totient. So 2 is a totient. Would you believe that 90 is ***not*** such a number, so it's a nontotient? I'm sure that that is the piece of information that has been missing from your life and that you are now a complete person!

If not, you might like to know that the famous mathematician Paul Halmos died at age 90 in 2006 (see http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Halmos.html); that the Interstate 90 is a major highway that runs across the north of continental USA from Washington to Massachusetts (see http://www.google.co.nz/images?rlz=1T4SUNC_enAU357AU360&q=interstate+90+map&um=1&ie=UTF-8&source=univ&ei=RZqITNf0OYLQcaejuJ4I&sa=X&oi=image_result_group&ct=title&resnum=5&ved=0CDkQsAQwBA&biw=1239&bih=567 – a URL which is about as long as the highway itself); that XC is 90 in Roman; that there are 90° in a right angle; that the bases in baseball are 90 feet apart; and that 1223334444555556667666555554444333221 is not divisible by 90 but it is divisible by 9.

Some thoughts on decimals

In the last newsletter I set out to see if there were any holes in the decimal numbers in the same way as there are holes in the fractions. Unfortunately, by the time I had begun to appreciate what the decimals were and how they were constructed I ran out of time for such esoteric things as holes in lines.

But let's go back to the fraction line and recall what was causing holes to appear there. It appears that $\sqrt{2}$ was the root of the problem. I didn't actually show that this was the case because it requires quite a subtle, but nice, proof. If you want to see that proof though, you can do worse than look up http://en.wikipedia.org/wiki/Square_root_of_2#Proofs_of_irrationality and follow the steps. It's one of the classic proofs in mathematics and goes back at least to Euclid's time.

So can $\sqrt{2}$ be written as a decimal? If so, I can plug the hole in the fractions when I walk along the decimals; but maybe before that I should make sure that any fraction can be written as a decimal. I think that that's straightforward though. I'll do something that is bad mathematics – I'll give a proof by example (when I should be proving it for ***all*** fractions). My excuse this time is that I'm only trying to show you that every fraction is probably a decimal. I won't actually prove that they all are (though they are!).

Doing it for an example is easy. Let me take 3/8. And now some long division is in order.

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   0.375
8 |3.000
   2 4
     60
     56
       40
       40
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So we have shown that at least one fraction is a decimal as 3/8 = 0.375. But we can do this with any fraction. Let's think about 3/7. Again we'll get 0. ... and then long division will get us a first decimal place, then a second, and so on. The interesting thing to note with 3/7 is that when we do the long division we have a series of subtractions. The result of these is always less than 7. (With 3/8 the subtractions resulted in first 6, then 4, then 0 – which I forgot to put in on the long division above.) That means that if we do 7 or more of these subtractions we have to come back to a result that we've already had. That means that the fraction will cycle round. Check out 3/7. You'll see that it gives you the decimal 0.428571428571428571428... Let me warn you not to do this on your calculator. If you do the blessed little machine will truncate the answer and you won't see the decimals go round and round the 428571 circle.

It turns out that all fractions do this circling round, even 3/8. The thing with 3/8 is that it goes 0.3750000000... You see it's circling round 0. If you start with any fraction you'll get one of these circling round decimals, they either circle around a fixed sequence of numbers or they circle round zero. But even more than that, if you start with a circling round decimal you can be sure that it's a fraction. I'll duck this proof too but refer you to http://en.wikipedia.org/wiki/Repeating_decimal if you are keen to see how it works.

The upshot of this then is that the fractions sit somewhere in the decimals since every fraction is a certain type of decimal. The question is though, is every decimal a fraction? If it is, then there must be a hole in the decimal line. If not, there might not be a hole.

One way to find another decimal that is not a fraction is to invent one that doesn't circle around. The simplest one I can think of is $N = 0.1011011101111011111011111101111111 0...$ The idea is to take first one 1 and then put in a zero, then take two 1s and then put in a zero, and so on and so on. So at some point there will be 195 ones followed by a zero and then 196 ones followed by a zero and so onward and upward. This number is clearly a decimal, that happens to go on for ever (as does 3/7 as a decimal) but never repeats. Intuitively, any bit that you thought might repeat would have a biggest string of ones with a zero front and back. But that biggest string of ones can never occur again as the strings of ones keep getting bigger. So N isn't a fraction.

Now that doesn't tell us that the decimal line hasn't got any holes in it though. For example, it's just possible that √2 may not be expressible as a decimal. In that case, the decimal line would share one of the holes of the fraction line. We know that √2 isn't a fraction, so if it can be written as a decimal it won't circle around. But can it be written as a decimal? I guess that you can get to it by making successive approximations. First you can calculate that $1.4^2 < 2 < 1.5^2$, so √2 lies between 1.4 and 1.5. The next step is to experiment with the squares of 1.41, 1.42. Since $1.41^2 < 2 < 1.42^2$, √2 lies between 1.41 and 1.42. You could keep playing this game for as long as you wanted to, getting a decimal that got closer and closer to √2. Imagining someone with an infinite amount of time, would lead you to a decimal for √2.

However, √2 isn't the only hole in the fraction line. If you take any whole number that isn't a perfect square, its square root isn't a fraction. But it is a decimal though.

In the last article I didn't tell you about all the other holes in the fraction line. For instance, π isn't a fraction so it is responsible for yet another hole in the fraction line. And we can get it in decimal form to whatever accuracy we desire. If you don't believe me try http://newton.ex.ac.uk/research/qsystems/collabs/pi/ . So although π is responsible for a fraction hole it isn't up to providing a hole in the decimal line.

Just to blow your students away, you might mention to them that the decimals are not just a bigger set than the fractions because we can find one or two odd numbers that are decimals but not fractions. There are in fact an infinite number of fractions. There are, clearly, an infinite number of decimals. It turns out that the decimals are a different type of infinity from the fractions (see http://en.wikipedia.org/wiki/Infinity). So the weird truth is that there are more holes in the fractions than there are actual fractions!!!! Don't you just love numbers?

Oh, and while I think of it, if you are trying to compare two fractions for size, as we have done in earlier newsletters, turning them into decimals is often a useful way to make the comparison. Which is bigger, 37/61 or 111/184? Well, the first starts off as 0.606557 and the second starts off as 0.603260. So 37/61 is the bigger. Anyone who's a whizz with the Numeracy Projects, though, should be able to find a simpler way of showing this.

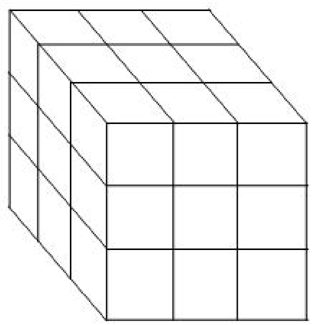
Problem Solving

These problems come from the high achieving students section of the website:
http://www.nzmaths.co.nz/activities-students

Years 1-3

I'm thinking of a 3-digit number. If I add its three digits together I get 26. What numbers could I possibly be thinking of?

Years 4 - 6



Tom was building with 20 red cubes and 7 white ones. All his cubes were of the same size. He made a larger cube using all of these smaller cubes. Part of the surface of the larger cube was red and part was white. What is the smallest fraction that could be white?

Years 7- 9

The Post Office down the road has run out of all stamps except 3c and nc stamps, where n is any whole number. They have a large amount of each of these. The post master isn't too worried though. He tells the person on the counter that he can make up every amount from (2*n* – 2)c onwards. Is that right? Can you justify it or can you find an amount of postage bigger than (2*n* – 2)c that the Post Office can't make up?

Answers to Problem Solving

Years 1 - 3

First we need to know what the three digits could be. What three digits when added together give a total of 26?

If one of the digits is 7 then the other three add to 26 – 7 = 19. But 2 × 9 = 18, so no two digits can add to 19. So none of the original digits is 7. In fact none of them is smaller than 7 for the same reason.

They can't all be 9 (3 × 9 = 27 > 26), so I must have one 8 and two 9s.

So, what numbers can be made up with one 8 and two 9s? There are just three of them. They are 998, 989, and 899.

Years 4 - 6

Tom needs to get as few of the white cubes on the surface of the bigger 3 × 3 × 3 cube as possible. Now it is possible to 'hide' one of the cubes in the middle of the 3 × 3 × 3 cube. Then there are six left. It is possible to make sure that only one of the faces of the white cubes is on the surface of the bigger cube. This can be done by putting the white cubes in the middle of each face.

So we have 6 white faces exposed on a 3 × 3 × 3 cube. Such a cube has 6 × 3 × 3 little cube faces exposed. This is a total of 54. So the smallest fraction of white that Tom can make is 6/54 = 1/9.

Years 7 – 9

It would be a good idea to test the post master's conjecture out by trying some simple values of n, say $n = 5$ and 7. For $n = 5$ you should be able to show that everything from 8 = (2× 5 – 2)c on can be made up. For $n = 7$ you can show that everything from 12 = (2 × 7 – 2)c on can be made up. So those two cases make it seem as if the post master is right.

Now try $n = 6$. With some quick calculations you will find that you can only make up multiples of 3. If $n = 6$, $2n – 2 = 10$. And you certainly can't make up 10c postage with only 3c and 6c stamps.

Classroom section

Poster Project

Producing a maths poster provides students with the opportunity to revise, clarify and share their learning. If your whole class is going to take part in the poster project it is best to select a topic that can be divided into different aspects so each pair of students can produce a different poster.

Suggested Topics:

- Addition strategies, subtraction strategies, multiplication strategies, division strategies
- Types of transformations, types of graphs, how to measure areas and volume of shapes

Suggested Content Requirements:

- Title
- Teaching about the concept
- Examples of the concept
- Use of the concept

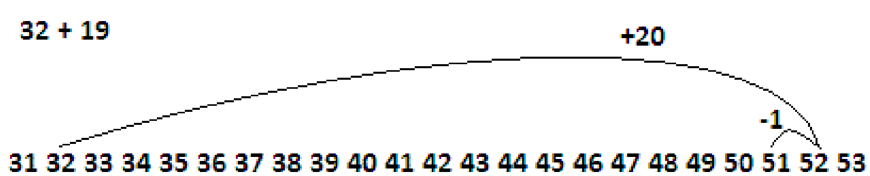
Addition with Tidy Numbers

A tidy number is a number like 10 or 20 that is easy to add to another number.

To solve 32 + 19 it is easier to make the 19 into the tidy number 20 because now I can add 36 + 20.

Since 20 is 1 bigger than 19 I need to take off 1 at the end. So 32 + 19 becomes 36 + 20 - 1. The answer is 51.

On a number line it looks like this:



Here are some more examples:
42 + 9 → 42 + 10 then - 1 → 52- 1 = 51
57 + 18 → 57 + 20 then - 2 → 77 - 2 = 75

Usefulness
This is a good strategy to use when one of the numbers is close to a tidy number.

Families

Many parents play number games with their children which helps them develop one aspect of their maths. Shape is another important area you can work with your children on. Here are a collection of activities relating to shape.

Construction with boxes

Collect old cardboard boxes (cereal, tissue, shoe, spice etc) for children to construct objects like buildings, robots, rockets, castles, etc . You will also need glue or tape, scissors, and if you want to decorate then paint or paper.

As you talk to your child about the construction talk about the faces, edges, corners of the boxes, the square and rectangular shapes of the faces.

Making shapes with play dough

Work with your child to make shapes with play dough.

Younger children can make flat shapes like squares, triangles, circles using a plastic knife. Talk about the number of sides.

Older children can make 3 dimensional shapes like cubes, spheres, cones, cylinders. Talk about the number of faces, edges and corners. Find out about the face of cross section by cutting the shape with a knife. For example if you cut through a cone you get a circle face, but if you cut from tip to base you get a triangle face.

Look for other items around the house that have two dimensional shapes or three dimensional shapes.

Tangrams

In newsletter 86 there was a copy of a tangram puzzle and pictures to make using the pieces. Talk to your child about how the lengths of different shapes match up, and how pieces fit together.

Origami

Making paper planes and hats are fun origami activities. Talk to your child about folding in half, the shapes (usually squares, triangles and rectangles) they see as them make it. Here is website with instructions to make a sailing boat.

http://www.activityvillage.co.uk/Origami%20Sailing%20Boat.pdf