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Rereading last month's editorial set me thinking. If all those topics like set theory, multibase arithmetic, vectors and so on that we taught during the 'modern maths' era were so fundamentally important to the curriculum and had to be taught to help children understand the structure of mathematics, why are they are no longer there? What does that mean? Are there practical reasons the topics were dropped? Was it because many of the then current crop of teachers were not brought up on modern maths and had difficulty grappling with the new philosophy? Or was it just a fad? A fashion - the result of filtration perhaps by educational bureaucrats from theories developed by those engaged in pedagogical research? Did we find that the ability of our pupils to do mathematics was not enhanced by all the 'modern' approaches? It would be logical then to drop them from the curriculum. Of course, this would need to be done discreetly, a little at a time - after all, we don't want to undermine the credibility of our curriculum designers. That seems to be what happened but we shouldn't assume that this implies our argument is correct. It could simply be that a 'swings and roundabouts' situation is taking place. A period of high creativity in any venture is often followed by one of 'back to basics' and curriculum design is not exempt from this. Still, we shouldn't assume that any particular curriculum is optimal. There may be no best set of mathematical ideas to teach children. What is appropriate for one generation may not be for the next. Where does that leave us? Well, we must keep looking at the needs of society, what mathematics is important now and for the near future. O.K., so it may be time to dump Euclidean geometry, Boolean algebra and long division but for goodness sake let's try and anticipate what will of greatest importance in the years to come.

Although we've included this quotation before in our news letter, it seems so appropriate and worth a repeat:

> We tend to think of multibase arithmetic as a fairly recent addition to the curriculum but my father did it from Hall and Knight's Elementary Algebra in the 1920s and his father did it from Isaac Todhunter's Algebra in the 1880s!!

R.A. Dear

H.G. Wells was awarded a D.Sc at the age of 78 for a doctoral thesis on 'The Quality of Illusion in the Continuity of Individual Life' and the Ayatollah Khomeini started his revolution to overthrow the pro-western Shah at this age. Dwight Eisenhower, Mahatma Gandhi, Charles Hermite, Pierre Laplace and Tiberius all threw in the towel aged 78.

Given the main topic of this newsletter, it's a little surprising that 78 isn't prime. Should we have waited until number 79 ?

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## What's new on nzmaths.co.nz

There are three new objects available in the Learning Objects section. These Learning Objects have been developed specifically for the nzmaths site and are not password protected so feel free to have a look. The new objects are:

1. Drive: Easy problems: which helps users solve proportional problems with distance and time
http://www.nzmaths.co.nz/LearningObjects/Drive/Drive.swf
2. Drive: Hard problems: A version with more difficult problem sets http://www.nzmaths.co.nz/LearningObjects/Drive2/Drive2.swf
3. Loans and Savings: which helps users find the nett balance of a loan account and a savings account
$\underline{\mathrm{http}: / / \mathrm{www} . \text { nzmaths.co.nz/LearningObjects/loans and savings/loans and savings.s }}$ wf

These Learning Objects are also available in Maori from this page: http://www.nzmaths.co.nz/maori/lo/default.aspx

There are also five new units of work available. They are:

- Mix Up 2
(http://www.nzmaths.co.nz/Number/Operating\ Units/LO/MixUp2.aspx)
- Loans and Savings
(http://www.nzmaths.co.nz/Number/Operating\ Units/LO/LoansandSavings.aspx)
- Angles (http://www.nzmaths.co.nz/Measurement/Angle/Angles.aspx)
- Drive (http://www.nzmaths.co.nz/Algebra/Units/drive1.aspx)
- Ratios (http://www.nzmaths.co.nz/Measurement/Volume/ratios.aspx)

This month we'd also like to invite YOU to submit material for the site. We are inviting submissions of activities to include in the Numeracy Planning Assistant, so if you have any short activities that you would like other teachers to have access to, please download the template from:
http://www.nzmaths.co.nz/numeracy/NumeracyPA/NumeracyPA_ActivityTemplate.doc Fill in the gaps and email it to andrew@nzmaths.co.nz. We will leave the name of the author at the bottom of all accepted activities.

## Why the Fascination With Primes?

Why do prime numbers intrigue so many people? Let's list a few examples of prime infatuation. The Endeavour that we have now every month has thought about them for at least 3 months in a row. We have just heard that the $44^{\text {th }}$ Mersenne prime was found on August $23^{\text {rd }}$ and the $45^{\text {th }}$ on September $6^{\text {th }}$ (see http://www.mersenne.org/prime.htm). Now those things have over 11 million digits so they aren't something that you decide to find when it's too wet to play croquet! And Mersenne must have been puzzled by them in the first place (yes there was a guy called Mersenne - see http://www-groups.dcs.stand.ac.uk/~history/Printonly/Mersenne.htm). But so have a gaggle, maybe lots of gaggles of people before and since. Maybe the GIMPS (Great Internet Mersenne Prime Search, http://www.mersenne.org/prime.htm ) website should have the answer. But they essentially say 'because they are there'.

But here's the thing, if you are one of those prime fanatics (or you would like to be or you know someone who is or would like to be) then you can actually get into the Mersenne Prime search business. So I'd better tell you first what Mersenne Primes are and if anyone is truly interested in them for reasons other than to notch up the next Mersenne Prime.

Let's go back to the first half of the Seventeenth Century when Marin Mersenne was working as a handsome monk somewhere in France (see the picture above from http://www-groups.dcs.stand.ac.uk/~history/Printonly/Mersenne.htm), maybe
 Bordeaux, and he decided that he wanted to find a formula for all of the primes. (If you go to this month's Endeavour you'll see that Euler, a century later was still trying to find such a formula so it was something that everyone wanted to do about then.) As it turned out, Mersenne didn't get one (or else Euler would have been wasting his time) but he did come up what something that was, and still is, pretty interesting - well to some people anyway. He hit upon

$$
\mathrm{M}=2^{\mathrm{p}}-1 .
$$

If you are prepared to do a bit of fiddling on the old calculator, you'll see that M is a prime for $\mathrm{p}=2,3,5,7,13,17$ and 19 . What's more there are a whole host of other p for which M is prime, well 45 in total - maybe that's not a whole host.

Mersenne found several but claimed a few that weren't but we won't hold that against him because he didn't have any electronic devices to help him out. So he did pretty well to get where he did. It probably meant though, that he spent many a sleepless night in his cell multiplying M's out and trying to factorise them.

Anyway, I have to say that he quickly noted that p had to be a prime itself otherwise M couldn't be. That's easy enough to see on some small examples. Suppose that p was 9 or 35 . Then
$2^{9}-1=\left(2^{3}-1\right)\left(2^{6}+2^{3}+1\right)$
and
$2^{35}-1=\left(2^{5}-1\right)\left(2^{30}+2^{27}+2^{24}+2^{21}+2^{18}+2^{15}+2^{12}+2^{9}+2^{6}+2^{3}+1\right)$.
You can use this trick on any composite number so, trust me, I'm a doctor, M can only be prime when p is. But, of course, if p is a prime that doesn't mean that M necessarily is. After all $2^{11}-1=2047=23 \times 89$. But if it ain't it ain't.

So all the Mersenne Prime hunters only have to go through all of the primes $p$ to test for the next M. That's not hard surely? The primes are pretty thin on the ground - again look at the Endeavour for the distribution of primes. Heck, there are only 4 between $10^{12}$ and $10^{12}+$ 100.

Unfortunately, and I don't really now how to tell you this if you don't already know it turns out that there are an infinite number of primes. They just go on and on and they never stop. Euclid knew this. He put it in his book The Elements. That book hasn't been out of print in over 2000 years (or do I exaggerate just a little?) so it must be right. And he proved it by this argument.

Suppose that there are only 53 primes. Then we can list them as $p_{1}, p_{2}, p_{3}, \ldots, p_{53}$. Now look at $N=p_{1} p_{2} p_{3} \ldots p_{53}+1$. What can $N$ be? If we are lucky it's a prime and our supposition was totally up the creek. So it's not a prime. So it must be divisible by some prime - all numbers are. Is it $\mathrm{p}_{1}$ ? Nagh. Is it $\mathrm{p}_{2}$ ? Nagh. Is it $\mathrm{p}_{3}$ ? Nagh. ... Is it $\mathrm{p}_{53}$ ? Nagh. So there must be a prime somewhere that is not on our list of 54 . Doesn't that show that there is an infinite number of primes?

Anyway even GIMPS admits that finding new Mersenne primes is not likely to be of any immediate practical value (however, see Afterthought). The search is primarily a recreational pursuit. However, the search for Mersenne primes has proved useful in the development of new algorithms, testing computer hardware, and interesting young students in maths. And if you want to be involved, and maybe get the $46^{\text {th }}$ Mersenne Prime, then GIMPS will tell you how.

## Booke Review <br> A Short Account of the History of Mathematics by W.W. Rouse Ball

First published in 1888, this historical summary of the development of mathematics, illustrated by the lives and discoveries of those to whom progress in the subject is mainly due, remains one of the most honoured histories. In it Rouse Ball treats hundreds of figures and schools that have been instrumental in taking mathematics forward, from the Egyptians and Phoenicians to such giants of the $19^{\text {th }}$ century as Galois, Hermite and Reimann. The semi-biographical approach, common in modern histories, gives a real sense of mathematics as a living science.

The book ran to a number of editions, the most accessible of which today is probably that by Dover published in 1960 and based on Rouse Ball's last, the fourth edition, which appeared in 1908.

I acquired my copy of the Short History, all 500 pages of it, when I first began teaching. Until that time I had no context for the subject I had studied and was now beginning to pass on to others. After reading the book I immediately read through it again making notes on who I considered to be the most important characters, summarising their main contributions. The two notebooks that I filled provided me with background and insight to my subject that became a major teaching aid for me over a great many years.

I have read, and in fact own, a number of other histories. Rouse Ball's Short History remains one of the most comprehensive, authoritative and accurate of them all. Even as an overview, it is a book that enables us to understand the development of one of the most difficult of all intellectual endeavours and the work of some the greatest contributors.

## Perfect Numbers

I'm surprised that we haven't said anything about perfect numbers in this newsletter before. I did a search and couldn't find anything. Actually perfect numbers is another one of these things that only exists to interest young people in maths - or is it?

Anway, what is it? Or rather what are they? Well they are kind of fascinating too. They are things like 6 . Now $6=1+2+3$. And 1,2 and 3 are all the factors of $6-$ if you turn an appropriately blind eye away from 6 itself.

Now $28=1+2+4+7+14$ is another perfect number. Again it's the sum of all of its factors except itself. So that's pretty much the definition of a perfect number.

The first four of these animals have been known since Adam was a boy. Well we certainly know that good old Euclid (that man again) in his Elements knew about perfect numbers. He even gave a way of finding them.

Here's his recipe. Take $1+2+4+8+16+\ldots$ until the sum is a prime number. (No, not primes again!) Then multiply that prime number by the last power of two that you added to get it. Funnily enough, that product is a prime. Here are two examples.
$1+2+4=7$, a prime; $7 \times 4=28$, a perfect number.
$1+2+4+8+16=31$, a prime; $31 \times 16=496$, a perfect number.
So now you can amaze your friends at the pub by coming up with a perfect number just as the All Blacks are about to score a try to win the next Bledisloe Cup.

But I'm afraid that it's at this point that good old Mersenne gets a jersey again. What Euclid was actually saying, in symbols, was this. If

$$
2^{\mathrm{k}}-1 \text { is a prime, then }\left(2^{\mathrm{k}}-1\right)\left(2^{\mathrm{k}-1}\right) .
$$

So look here. If you take a Mersenne Prime M , for some prime p , then $\mathrm{Mx} 2^{\mathrm{p}-1}$ is a prime. So maybe young Marin $M$ was only looking for Mersenne Primes because he wanted to find some more perfect numbers? Perhaps there is a use for these big primes after all.

Now I'm not sure that I want to ask this question, but what use are perfect numbers?

## And They Are Useful

I still can't answer the last question but I can tell you why you might want to have a big prime number sitting around or even two. And it's not just to find more perfect numbers.

It appears that factorising numbers is hard. Seriously. OK so you can all tell that 95 is divisible by 5 because it's got a 5 at the end. And 5 into 95 goes 19 so $95=5 \times 19$. But if I give you a number with a 100 digits it would take you more than a little time to factor. Actually it would take most people, even experts with computers a considerable time. And a form of cryptography is built around that difficulty. I won't go into the details here but there is a good chance that your bank, the FBI, and any other group that want to send messages over an open channel, uses something like the RSA form of cryptography. This depends on the fact that if you (let me assume that you are the enemy here) are faced with a number that is the product of two very large primes, you won't be able to factor it in a hurry. Actually by the time the message that you wanted to read has been decoded, nobody will be very interested in what it said anyway.

To find out more about this, look at http://en.wikipedia.org/wiki/RSA .

## Solution to August's Endeavour

There are 23 prime numbers in LIST 1, namely, $\{5,7,11,13,17,19,23,29,31,37,41,43$, $47,53,59,61,67,71,73,79,83,89,97\}$. LIST 2 has two extra, the primes 2 and 3.

No, it is not true that almost all primes are one more or one less than a multiple of six, although it seems that way just looking at LIST 1. What we can say is that for primes less than 100 almost all are one more or one less than a multiple of six. The higher we go the less dense prime numbers are. For example, there are 25 primes between 0 and 100, 21 between 100 and 200, 16 between 200 and 300 . There is not a steady decrease as we move higher up but a general trend. For example: Between 500 and 600 the number of primes has dropped to 14 but between 600 and 700 the number is 16 . So although the density of multiples of six remains constant as we move into higher numbers the density of primes generally decreases. There are only four, for example, between $10^{12}$ and $10^{12}+100$.

88, 96 and 98 cannot be expressed as the difference between pairs of primes from LIST 2 but this neither supports nor refutes the conjecture. Each of the three numbers can be expressed as the difference between two other primes. For example: $88=467-379,96=$ 479-383 and 98=577-479 but the original conjecture was about consecutive primes not any primes.

## This Month's Endeavour

$31,331,3331,33331,333331$, are all prime numbers. Does the pattern continue (a) at all ? (b) for ever?

Euler was very pleased with a formula he conjured up to produce primes. It is, in words, 'Think of a whole number and multiply it by one more than that whole number, then add $41^{\prime}$. In symbols this is $\mathrm{n}(\mathrm{n}+1)+41$ where n is a whole number. Does this formula always provide prime numbers? If not, what is the smallest value of n for which it doesn't?

## Solution to last month's Junior Problem

Last month's problem must have been the shortest that I've ever posed. How many squares are there on a chessboard if you don't just consider the 1 by 1 or 8 by 8 squares?

Our winner is Timothy Berry from Reporoa Primary. His solution is below.

The answer to this months junior problem is
This is how I did it!!!!!!!!!

Add together these squares $8 \times 8,7 \times 7,6 \times 6,5 \times 5,4 \times 4,3 \times 3,2 \times 2,1 \times 1$

You can follow this up by looking at any squares on the chessboard, even ones that you can't actually see without a bit of imagination. For instance, how many squares are there that have corners at the corners of chessboard squares but which don't have their sides parallel to the sides of the board?

## This Month's Junior Problem

To get this month's \$20 I'm staying with boards. But let's look at a rectangular board. How many rectangles are there (parallel to the sides of the board) on a $5 \times 6$ board? Does the picture help?


You might want to start with a $1 \times 2$ board and work up.

## Afterthoughts

What we have been talking about in the Solution to last month's Endeavour is the distribution of primes. Suppose that you wanted to know the number of primes less than or equal to some number n . Then it turns out that a not bad approximation is $\frac{x}{\ln x}$. So if you want to find the number of primes less than or equal to 100 , you'd get about $\frac{100}{\ln 100}$ which is about 21.7 - a bit out but the approximation does a lot better as n gets larger.

Now if you want a better approximation a little calculus comes in handy. Then try out $L i(x)=\int_{2}^{x} \frac{1}{\ln t} d t$. That's surprisingly accurate - check it out.

And if you have forgotten all the integration you ever knew, or if you never knew any in the first place, have a look at http://en.wikipedia.org/wiki/Prime_number_theorem, where they do all the work for you. That's a nifty graph they have produced.


Graph comparing $\pi(x)$ (red), $x / \ln x$ (green) and $\operatorname{Li}(x)$ (blue).
Oh, and why is the 45th Mersenne Prime smaller than the 44th?

*     *         * 

And just in case you thought that all mathematics was sewn up and that there was nothing more to do, we've given you two open problems here. First, find ALL Mersenne Primes and second find ALL perfect numbers.

