Newsletter No. 51 **March 2006**

It's amazing how the range of applications of Mathematics has widened over the last half-century. A couple of generations ago it was mainly mathematicians, physicists and engineers who had any great use for the subject, nowadays it is different. Any student aspiring to study in depth subjects such as Biology, Chemistry, Geography, Sociology, even Psychology would be unwise to omit a good grounding in Mathematics. Then there are relatively new subjects like Operations Research, Consumer and Applied Science, Computer Studies and Health Science which rely more than a little on an ability to think mathematically. Mathematical modelling permeates much of our thinking today. Politicians, meteorologists, architects, agriculturists, lawyers and historians, to name just a few, all make use of mathematical models both in their studies of past events and in their planning for the future.

We live in a changing world, one that presents us with an ever-increasing range of new problems. To solve them we need a wide understanding of current mathematical techniques and the creative ability to devise new ones when the need arises.

> *There are things which seem incredible to most people who have not studied mathematics.*
>
> **Archimedes**

Did you know that 51 is the smallest number which can be written with all the digits from 1 to 5 (without repetition) as a sum of primes: $51 = 2 + 3 + 5 + 41$? You might know that 51 countries originally signed the UN charter in 1945 and that Honoré de Balzac, Napoleon Bonaparte, Calamity Jane, Gordon of Khartoum, Ronnie Lane (of the Small Faces) and Jan van Eyck all died aged 51.

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What's new on nzmaths.co.nz

The 2006 versions of the Numeracy Project Books are now available in the Numeracy Project section of the site. The Planning Sheets and 'I Can' Sheets which were previously included in Book 3 are now held separately in the Numeracy Project section of the site.

Another Game

Over the last few newsletters I've talked about two combinatorial games – games where each player knows all of the rules. We looked at some noughts and crosses type games (September 2005) and the counting game '22' (November 2005). In all of these games, it is possible to say from the start whether the first player or the second player must win, provided that player plays 'properly'. Here I want to look at another such game and one that is very close to the '22' game.

I'll call this game, the '24 Game'. Put 24 blocks on the table and ask Alice and Blair to play. Each person on their turn may take 1 or 2 blocks. The winner is the one who takes the last block. As befits the age of chivalry in which they live, Alice always goes first. Does Alice or Blair always win?

The way to go about analysing these kinds of games, is to play with smaller numbers of blocks first. Below I have a put my results in a table.

1	2	3	4	5	6	7	8
A	A	B	A	A	B	A	A

By the entry A under 5, for example, I mean that A wins if there are 5 blocks on the table to start with.

It turns out that multiples of 3 are critical to this game. The wining strategy is always to take the number of blocks down by 3. For instance, if there are 24 blocks, Whichever number Alice takes (1 or 2), Blair takes enough for the number of blocks to go down by 3 (so 2 or 1). So it is Alice's turn when there are 21, 18, …, 6, 3 blocks. But at 3 blocks Alice can't win. If she takes 1, Blair takes 2; if she takes 2, he takes 1. In each case he gets the last block and wins.

So who would win if there were 25 blocks and how would the winner play? What about 26, 27, 39, 56, 299 blocks?

And when you have mastered that you might like to change the rules so that the players can take 1, 2, or 3 blocks when it's their turn. Who would win starting with 24 blocks then? What about 26, 27, 39, 56, 299 blocks?

Can you see where to go next?

Strings to 50

In February's newsletter, I started some people thinking about what could happen when you added together strings of consecutive numbers. The idea was to try to see what was going on when you added strings like 3, 4, 5, 6 and 89, 90, 91, 92, 93. For instance, is every number the sum of such a string?

From Marnie Fornusek of Rotorua I got a lot more than I expected. Here is part of what she sent in.

"The sum of an individual string is of the form 2n+1 for strings of two numbers, 3n+3 for three numbers etc where n is the starting number for the string.

Form	Possible sums	All strings of a given type
2n+1	3, 5, 7, 9, ….	1+2, 2+3, 3+4, 4+5,…
3n+3	6,9,12,15,…	1+2+3, 2+3+4, 3+4+5,…
4n+6	10, 14, 18, 22,	1+2+3+4, 2+3+4+5, 3+4+5+6
5n+10	15, 20, 25, 30	1+2+3+4+5, 2+3+4+5+6
6n+15	21, 27,…	1+2+3+4+5+6, 2+3+4+5+6+7,
7n+21	28, ….	1+2+3+4+5+6+7

"I then went through and marked which of the numbers 1 – 30 had strings of what type e.g. 2 means it was the sum of a 2-string etc. The numbers without any strings (marked in grey) were 4, 8, 16 then to continue the pattern 32, 64 etc. Conjecture: numbers without string are of the form $4 \times 2^{m-1}$.

I've highlighted in yellow those that were the sum of 1 string."

1	**2**	**3**	**4**	**5**
		2		*2*
6	**7**	**8**	**9**	**10**
3	*2*		*2, 3*	*4*
11	**12**	**13**	**14**	**15**
2	*3*	*2*	*4*	*2, 3, 5*
16	**17**	**18**	**19**	**20**
	2	*3, 4*	*2*	*5*
21	**22**	**23**	**24**	**25**
2, 3, 6	*4*	*2*	*3*	*2,5*
26	**27**	**28**	**29**	**30**
4	*2, 3,6*	*7*	*2*	*3,4,5*

Looking at Marnie's table let's think about those that are not the sum of any string. To me her conjecture looks good but I'd like to add the numbers 1 and 2 because they fit the pattern too. This way we know for sure that 1, 2, 4, 8, and 16 are not stringable – can I call these ***0-stringable***? And Marnie's guess of 34, 64, and so on, sounds as if it's on the right track. If I can just push Marnie's guess a little, we get

Marnie's Conjecture: 0-stringable numbers are of the form 2^{m-1}. (See Comment below.)

Let's now look at Marnie's yellow numbers, ***the 1-stringable numbers***. What do we find? Well, 3, 5, 6, 7, 10, 11, 12, 13, 14, 17, 19, 20, 22, 23, 24, 26, 28, 29. That is a nasty jumble of numbers but it has a very interesting sub set. And that is 3, 5, 7, 11, 13, 17, 19, 23, 29. These are all prime numbers. Does that suggest the next conjecture?

Conjecture 2: All odd prime numbers are 1-stringable.

But what are the remaining 1-stringable numbers? So far we have 6, 10, 12, 14, 20, 22, 24, 26, 28. If we factorise these we get 2 x 3, 2 x 5, 2 x 6, 2 x 7, 2 x 10, 2 x 11, 2 x 12, 2 x 13, 2 x 14. That's nearly 2 x (a prime); but not quite. How can we get all these numbers to fit into one set?

One final thing that is worth noticing is that we can get a formula for the sum of a string. I won't go back over this again here but I will send you to another part of this web site. Have a look at InfoCentre, Seminar, Number – Gauss' Trick. At the very end you'll see that if you sum an r-string starting from n, you'll get

$(2n + r – 1)r/2$.

You might find that there is some interesting material in that Seminar that you could even use in your class. And you can actually use that formula to prove a few things – see Comment (3) below.

So what numbers are 1-stringable? And what numbers are 2-stringable? The line is still open for comments. And conjectures are fine; I don't need proofs.

Comments:

(1) If you think about it, the Conjecture even works for 1. If you let m = 1 you get 2^0, and $2^0 = 1$.

To see why this convention is used divide 2^3 by 2^3. This certainly gives 1.

But another way of looking at it is to use one of the index laws. 2^{34} divided by 2^{11} is 2^{23}. This comes about because when you are dividing one of these by another you simply subtract the indices: 34 – 11 = 23. So $2^3 \div 2^3 = 2^{3-3} = 2^0 = 1$.

(2) It's not too hard to show that Marnie's sums for 2-strings, 3-strings, and so on are correct. Let's do it for 2-strings. Now they start with n, so the next number has to be n + 1. Now n + (n + 1) = 2n + 1.

And for 3-strings you get $n + (n + 1) + (n + 2) = 3n + 3$.

The rest work by simply adding them all the terms up too.

(3) These conjectures are not easy to prove but we'll prove Conjecture 2.

Let p be a prime and let p be the sum of an r-string. So from what we did above we know that

$p = (2n + r - 1)r/2$ or $2p = (2n + r - 1) r$.

But the left side just consists of two primes and r is a factor of the right side. So $r = 1, 2, p$ or $2p$.

If $r = 1$, we're not interested because in this overall game we are never interested in strings of just one number.

If $r = 2$, then $2n + r - 1 = 2n + 1 = p$. So $n = (p - 1)/2$ and $n + 1 = (p + 1)/2$, and that gives us a 2-string.
If $r = p$, then $2n + p - 1 = 2$. But $p \geq 3$, so n has to be zero or negative. This is not possible.

If $r = 2p$, then $2n + p - 1 = 1$. This is again not possible.

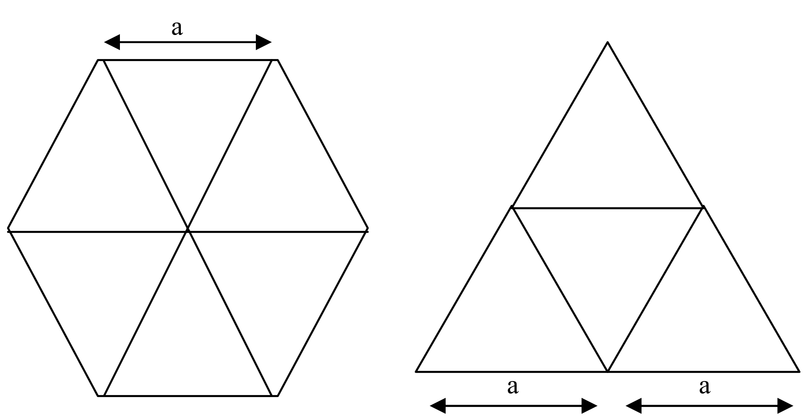
Having looked at all cases, p can only be written as a 2-string and so odd primes are always just 1-stringable.

Solution to February's problem

In February's problem we were told that an equilateral triangle and a regular hexagon had equal length perimeters and we were asked for the ratio of their areas.

Derek Smith of Lower Hutt sent in this very complete solution looking at the problem from three ways.

Geometrically



6 equilateral triangles (sides of length ‘a’) fit inside the hexagon as opposed to 4 (sides of length ‘a’) in the equilateral triangle as the perimeters are the same i.e. Perimeter = 6a. Therefore ratio hexagon : equilateral triangle is 6:4 simplifying to 3:2

Algebraically

(i) Using trigonometric ratios:

Area of hexagon is:

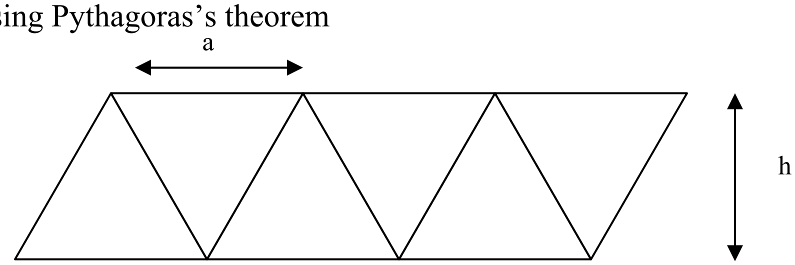
$$6x \ ½ \ x \ a \ x \ a \ x \ \sin 60^{\circ} = 3a^2 \sin 60^{\circ} = 3a^{2\sqrt{3}}/_2$$

Area of equilateral triangle is:

$$½ \ x \ 2a \ x \ 2a \ x \ \sin 60^{\circ} = 2a^2 \sin 60^{\circ} = 2a^{2\sqrt{3}}/_2$$

Therefore ratio hexagon : equilateral triangle is $3a^{2\sqrt{3}}/_2$: $2a^{2\sqrt{3}}/_2$ simplifying to 3:2.

(ii) Using Pythagoras’s theorem



Area of the rearranged hexagon = 3 x a x h = 3ah and $h = \sqrt{}(a^2 – (½ a)^2)$, yielding the area = $3a^{2\sqrt{3}}/_2$

Similarly by the same reasoning for the equilateral triangle = 2 x a x h = 2ah and $h = \sqrt{}(a^2 – (½ a)^2)$, yielding the area = $2a^{2\sqrt{3}}/_2$

Again the ratio hexagon : equilateral triangle is $3a^{2\sqrt{3}}/_2$: $2a^{2\sqrt{3}}/_2$ simplifying to 3:2.

This Month's Problem

This month's problem is related to the one that we asked last month. If the sum of the lengths of the edges of a cube is equal to the sum of the lengths of the edges of a (regular) tetrahedron, which has the largest surface area, the cube or the tetrahedron, and what is the ratio of these surface areas?

We will give a book voucher to one of the correct entries to the problem. Please send your solutions to derek@nzmaths.co.nz and remember to include a postal address so we can send the voucher if you are the winner.

Solution to February's Junior Problem

Allan had batting averages before and after Christmas of 70 and 30, while Marty had respective averages of 50 and 20. Is it possible that they could end up with the same overall average for the season? If so, how? If not, why not?

If we could fiddle things so that the same overall average was 35, then we'd be done.

Let's start with Marty. It turns out that the average of 50 and 20 is 35. This means that provided he is out the same number of times before and after Christmas we can wangle an average of 35. So let him be out twice in each half of the season. To get his first average of 50, he'd have to have scored 100 runs (100 runs for twice out gives an average of 100/2 = 50). To get his second average of 20 he'd have to score 40 runs.

His season average would then have to be 140/4 (number of runs over number of outs) = 35 as we wanted.

But Allan is more complicated and this needs a lot of fiddling (unless you know algebra). One way to get Allan an average of 50 is to let him score 70 runs before Christmas for once out (70/1 = 70); and to let him score 210 after Christmas for 7 times out. This gives a total average of 280/8 = 35.

I'm sorry to say that we didn't get a single answer to this problem.

This Month's Junior Problem

Let's keep on the track of averages but make it, hopefully a little easier than last month.

During term 1, Alice averaged 80 over all of her tests. During term 2 she averaged 70 for all of her tests. Can you find a way that her overall average for the first two terms was 75? Is it possible that she could have averaged 90 for the first two terms?

To put a claim in for the $20 book voucher, send your solution to Derek at derek@nzmaths.co.nz. It would help to have your school address, and teacher's name added to the message. Make sure that your teacher or parents know that you have sent in a solution.