**Newsletter No. 37** **September 2004**

In our June 2004 issue we listed some definitions given by students of the word decimal. Some of them were bizarre and we reflected on whether this was due to the fact that our maths students are rarely asked these days to closely define the concepts they use. If they can't define them, one wonders, do they fully understand them? If it is simply that the exercise of careful definition is unfamiliar surely much could be gained by looking at this with our pupils.

The way I cover the topic is to ask pupils to carefully define a particular word that we use regularly in our lessons. Examples have included: angle, area, perimeter, volume, circle, ratio, fraction and statistics. I give them one word at the beginning of class and the pupils write their definitions on small pieces of paper that are then collected in. I might do this once a week, for example. It provides a good way of focussing attention at the start of a lesson. After the papers have been collected the responses are considered for discussion – that way each attempt remains anonymous. A consensus is reached which the class (and teacher) feels most closely defines the word under consideration.

Here are some responses received to the question, 'What is a circle?'

*A round thing like a flattened tennis ball*
*A shape with no sides*
*A shape like a hole*
*A shape with no ends*
*A line which finishes where it starts*
*A round line with no angles and corners*
*A closed curve with a diameter and circumference*
*A round sphere like a hula-hoop*

Why not give it a try - and be prepared for surprises!

And while we're about it I just want to say a bit more about last month's lead topic – Hardy's Conjecture. Recall he thought that there might be an infinite number of pairs of primes that are two apart. Then last year, along came Ben Green and Terence Tao to show that much more than this was true. In fact there are infinite sets of primes in arithmetic progression. We talked about Terence Tao's connection with the International Mathematical Olympiad. It turns out that Ben Green won a couple of medals at the IMO too and he has also coached recent British IMO teams.

And on the subject of IMOs, if you know some bright young secondary students who you think might benefit by some IMO training, you should get them in touch with the September problems that the NZIMO Committee are posing about now. They will appear on the New Zealand Association of Mathematics Teachers web site pretty soon (www.nzamt.org.nz). Able maths students should try these and send them in to Alan Parris at Linwood College, Christchurch to be marked. The best 25 students who reply will be chosen for the NZIMO Training Camp in January next year.

## What's new on the nzmaths site this month?

There are two new units, Te Paemahana, and Te Whakaari Raraunga, available in the Te Poutama Tau section of the website this month.

Book 9 of the Numeracy Project series (Teaching Number through Measurement, Geometry, Algebra and Statistics) has been posted in the Numeracy Project materials section of the site. The Material Masters for this book are nearly completed and should be available on the site this month.

In the last month we have transferred the hosting of the nzmaths site to a new company due to some inconsistencies in the performance of our existing hosting. We are confident that our new host will provide more reliable service than the old one did.

## What is the best order? by Brian Bolt

In my years as a school teacher I came across several situations where I was expected to put individuals or teams in their order of prowess. Here I would like to share the problems involved with two of these and let you draw your own conclusions. In my first term, as form master of 30 boys, I became aware of the panic setting in among the staff towards the end of term as the deadline for completion of report writing approached. For the form masters the computation of the form order was a real headache. Someone in their wisdom had decreed that in each subject a percentage should be given to each boy for their term's homework and averaged with their exam percentage before being multiplied by the number of periods the subject was taught in a week. These were then aggregated across the subjects for each boy in the form and the boys put in the order of their grand totals. This was before the days of calculators so you can understand the concern of many teachers. Their one aid, kept in the staff room, was a kind of slide rule to convert term work totals to a percentage. (Wise staff planned their coursework to be out of 100 in the first place!) What an horrendous system!

A little thought will soon make you realise how advantageous this was for the boys good at maths who could score very high percentages which got multiplied by 5 for the periods taught per week, compared to historians who did well to get 65% and then only had it multiplied by 2.

It took me a year to convince the staff that other easier ways of arriving at a form order were just as valid, although not necessarily giving the same result. I recommended that each subject teacher should put the boys in a pecking order for their subject and these orders be aggregated for each boy with the smallest total corresponding to the top of the form and so on. This overcame the problems of maths having a wide spread of marks compared to English and History where traditionally the spread is very small. It also put subjects with a small teaching time on a par with those having many periods, which didn't go down well with everybody. But overall the staff felt the final order corresponded better to their intuitive feel of the group. This method is very like that used in World Cross Country Races where the position of each runner as he/she finishes is aggregated for each team of six runners. To illustrate how difficult it is to arrive at a fair result, let alone a best or correct result I have made up a table giving the exam marks of five candidates who took six subjects and consider two ways of ordering them:

i) aggregating the percentages;
ii) aggregating the positions of the candidates in each subject

| | Maths | Hist | Eng | Sci | Geog | RE | Total |
|---|---|---|---|---|---|---|---|
| Andrea | 99 | 50 | 45 | 84 | 62 | 50 | 390 |
| Brian | 25 | 64 | 65 | 40 | 65 | 60 | 319 |
| Chao | 92 | 53 | 51 | 85 | 58 | 49 | 388 |
| David | 71 | 60 | 49 | 55 | 66 | 55 | 356 |
| Erin | 65 | 65 | 60 | 51 | 49 | 58 | 348 |

By simply aggregating the subject marks the pecking order becomes:
1. Andrea; 2. Chao; 3. David; 4. Erin; 5. Brian.

Now consider what happens if we look at their subject orders:

| | Maths | Hist | Eng | Sci | Geog | RE | Total |
|---|---|---|---|---|---|---|---|
| Andrea | 1 | 5 | 5 | 2 | 3 | 4 | 20 |
| Brian | 5 | 2 | 1 | 5 | 2 | 1 | 16 |
| Chao | 2 | 4 | 3 | 1 | 4 | 5 | 19 |
| David | 3 | 3 | 4 | 4 | 1 | 3 | 18 |
| Erin | 4 | 1 | 2 | 3 | 5 | 2 | 17 |

This time the order is a complete reversal of the former one:
1. Brian; 2. Erin; 3. David; 4. Chao; 5. Andrea.

Given these marks how would you have ordered them? A statistician might standardise the raw marks before aggregating and with modern computers this is not difficult. But would you have faith in the outcome?

My later experiences as chief external examiner of a multidisciplinary BEd degree course highlights the problems of trying to decide on the classification of an individual's degree where tutors from disciplines with very different traditions of marking tried to reach a fair conclusion.

But back to my teaching days, where as a keen sportsman I was rapidly co-opted to help run the cross country and athletics teams. By chance or design I found that most of the teachers responsible for athletics in our opponents' schools were mathematicians and the tradition had developed that the home team should be allowed to decide on the way points be awarded for each athletic event. It soon dawned on me how important this was for the overall result of a match, and the following example will help illustrate this.

In an athletics match between Athlone Academy and Barchester College there were just 8 events. Two competitors from each school took part in all the events and the resulting positions obtained by the schools in the match are given in the table below:

| Position | 1st | 2nd | 3rd | 4th |
|---|---|---|---|---|
| Athlone | 6 | 0 | 4 | 5 |
| Barchester | 2 | 8 | 4 | 2 |

Who do you think should be declared the match winners?
In inter-school matches of this kind various scoring systems are used and three such are illustrated in the following table where the points given for a particular position in each event are shown.

| Position | 1st | 2nd | 3rd | 4th |
|---|---|---|---|---|
| System x | 5 | 3 | 2 | 1 |
| System y | 3 | 2 | 1 | 0 |
| System z | 6 | 3 | 2 | 1 |

Each of these systems have plausible arguments to support them, and you may have your preference.

Work out the points total for each team using each of the scoring systems. Which scheme would you back if your team had some outstanding individuals but little depth as against a team with few likely winners but strength in depth?

Rugby fans must often reflect on the result of matches won by an outstanding kicker whose penalty kicks dominate a match when what they want to see are more tries. Should more points be given for a try? It is interesting to speculate on how this would change the result of a match, but such a change in the weighting of the points would almost certainly change the strategy employed in the game so cannot really be used after the event.

The approach to football (soccer) league games would arguably be changed for the better if points were added for goals scored. Investigate the way points are awarded in other sports such as squash or basketball and the effect on the games concerned.

[Footnote: This article is a practical example of a result by the Noble Prize winning Economist, Arrow. He made three perfectly reasonable assumptions about voting systems. On the basis of these assumptions he was able to prove that there is no perfect voting system. That is, no matter whether you used first-past-the-post, MMP, or any other system, there would always be a situation where, for a given set of votes, a given candidate would be unfairly elected. The only time this result is not true is if there are only two candidates. In that case the first-past-the-post scheme satisfies all three assumptions.

If you think of the systems that Brian used above as voting systems (rankings) that rank the candidates (students/athletes), then you can see how his 'voting' schemes favour different sets of votes (student marks/positions in races).]

## More on Magic Squares

A year ago (September 2003) Brian Bolt wrote something about 3 x 3 magic squares. If you remember, a 3 x 3 magic square consists of nine numbers place in a 3 x 3 array, so that the sum of each 3 horizontal, vertical and diagonal numbers is the same. Just to refresh your memory here is a well-known 3 x 3 magic square. Its magic number is 15, that's the number any 3 horizontal, vertical and diagonal numbers add up to.

| | | |
|---|---|---|
| 6 | 1 | 8 |
| 7 | 5 | 3 |
| 2 | 9 | 4 |

Although that's the most well-known 3 x 3 magic square, it's not the only one. In fact there are an infinite number but don't worry about that. What you might want to worry about though, is how to make up some more. That turns out to be easy and here's how.

Suppose that we want to make a 3 x 3 magic square with magic sum of 21 and we don't much care what numbers we use. So choose your favourite three numbers, say 5, 6, and 7. Put these numbers in the three places shown below.

| | | |
|---|---|---|
| 5 | 6 | |
| | 7 | |
| | | |

Now 5 + 6 + 10 = 21, so we can fix up the first row by putting 10 in the 'open' position. Then 5 + 7 + 9 = 21, so 9 has to go in the bottom right hand position. And 6 + 7 + 8 b= 21, so 8 goes in the middle of the bottom row. So far we've got to the stage below.

| | | |
|---|---|---|
| 5 | 6 | 10 |
| | 7 | |
| | 8 | 9 |

But now we can put 2 in on the middle right position and 4 in the bottom left hand position. Once that's been done, it's easy to finish the whole square off to give:

| | | |
|---|---|---|
| 5 | 6 | 10 |
| 12 | 7 | 2 |
| 4 | 8 | 9 |

You might just check that we have in fact produced a 3 x 3 magic square. What's more you might now choose your own magic sum and your own three starting numbers and so produce your own 3 x 3 magic square. Now in this exercise you don't need to feel constructed to whole numbers, or positive numbers, or even fractions. Let your hair down and see where you end up.

To give us some language to communicate easily, we'll call the three initial positions we used in the magic square above (top left, top middle, and middle), ***generating positions***. This is because we can generate the whole of the magic square from these positions once we have chosen a magic sum.

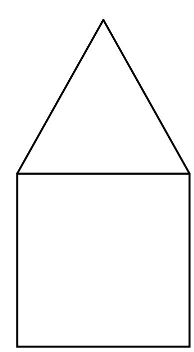
In the magic square above we used three generating positions. This raises several questions.

1. Are these, to within symmetry, the only three generating positions?
2. Are there two generating positions?
3. Are there four generating positions that do not contain three generating positions?
4. Are there five generating positions that do not contain four generating positions?
5. What is the largest number of generating positions that does not contain a smaller number of generating positions?
6. What is the smallest number of positions that does not contain a smaller number of generating positions?
7. Are there three positions that are not generating positions?
8. Are there four positions that are not generating positions?
9. What is the largest number of positions that is not able to generate a magic square?

And when you have all that sorted out, you might look at generating positions for 4 x 4 magic squares. We'll come back to this again next month.

## Solution to August's problem

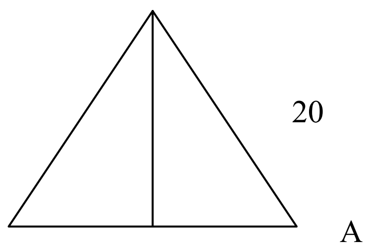
On a farm in South Otago there is an unusual storage shed. It is 20 metre wide and shaped like a cube with a pyramidal roof. Each face of the pyramid is an equilateral triangle. What is the area of the roof?



We had an unusually large number of entries for this problem and it really was very difficult to pick a winner. In the process of looking even harder, we discovered that there are at least three ways to find the area of an equilateral triangle of side length 20 m. We go through these below.

Method One: Use Pythagoras' Theorem. The basic idea here is that the area of a triangle is half the base times the height. We know the base is 20 m but to find the height we have to use Pythagoras. It is the unknown side in a right angled triangle where the other two sides are 10 and 20. Pythagoras gives the height is $\sqrt{30}$. This leads to an area of ½ x 20 x $\sqrt{300}$. Four of these give the total area required of approximately 693 $m^2$.

Method Two: Use trigonometry. The method here is essentially the same as the first method but you can find the height using trigonometry.




The height is 20 sin A. Since the triangle is equilateral, A = 60º, so the height 20 sin 60º. The rest follows as in Method One.

Method Three: Use Heron's Formula. This method is quite different from the other two in that the area is not fund by using 'half the base times the height' but by using Heron's Formula. This is that the area is given by $\sqrt{\{s(s-a)(s-b)(s-c)\}}$ where the three sides of the triangle are a, b and c and s is half of the perimeter (s = ½(a+b+c)). Since the equilateral triangle involved has a = b = c = 20, then its area is $\sqrt{\{30 \text{ x } 10 \text{ x } 10 \text{ x } 10\}}$. The total area follows again by multiplying by 4.

Because only one person came up with this last method, and because the other methods were given by at least two people, we've given this month's vouchers to Rachael Hoddinott from Christchurch.

## This Month's Problem

Ignoring zero, one is the smallest whole number that is simultaneously a perfect square, cube and fifth power. What is the next smallest whole number with this property?

We will give a petrol voucher to one of the correct entries. Please send your solutions to derek@nzmaths.co.nz and remember to include a postal address so we can send the voucher if you are the winner.