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Newsletter No. 33 **May 2004**

Deborah Lambie's delightful offering on the Twelve Days of Christmas in our last issue set me thinking that perhaps we don't offer our pupils enough opportunities to be completely creative in mathematics. I don't mean we don't encourage their creativity, we do that, for example, every time we make use of material on the nzmaths website. Problem solving exercises encourage us to search for strategies then maybe, when we've found one, look for another that is more efficient. We're certainly making creative use of our knowledge when we do that. What I mean is that perhaps we don't offer enough open-ended exercises to our pupils. We don't let them look at problems for which there is no particular solution but rather a range of different options subject to their imagination. Simple examples might be the one offered in last issue's editorial on deciding the worth of each of the whole numbers, another, and, also from our last issue, Reg Alteo's 'Twoons' which inspired Deborah Lambie's epic poem.

A thought that comes to mind might be deemed a cross-curricular exercise. Why not hold a 'balloon' debate with your maths class. If you haven't met one before, they go like this. Allocate some mathematical item or concept to each of, say, half-a-dozen pupils. For example, if you want to cover a bit of basic geometry you might use line segment, triangle, square, rectangle, hexagon, circle (more ideas below in 'Afterthoughts'). The idea is that each student has to justify why s/he remains in the balloon. Someone arguing for the triangle might say - it's the polygon with the least number of sides, all triangles tesselate, the right-angled triangle is a personal friend of Pythagoras, and so on. The scenario is that the balloon is losing height and someone has to jump out to save the others. After everyone has given reasons why they (i.e. their shape, number, whatever) should be allowed to remain in the balloon the rest of the class vote to see who should jump. The balloon is then deemed to be sinking again and the remaining people on board get another thirty seconds or so to justify why someone else should jump next. The process continues until only one remains. Of course, teachers can organise the time allowances and so on to suit their particular situation. The exercise gets pupils really thinking about mathematical concepts - and that's what we want, isn't it?

This month we would like to thank John Stillwell, Andy Begg, and Cathy Walker for their articles. In some way they all help us to extend our mathematical horizons.

The desire to explore marks out the mathematician

W. W. Sawyer

Diary Dates

Maths week on 8-15 August is still a way off - but getting closer. Initial information about Maths Week 2004 and resources from previous years are available on the NZAMT site: http://www.nzamt.org.nz/

Putting Things In Perspective by John Stillwell

It's a common saying, but an odd one when you come to think about it. To put something in perspective is supposed to mean seeing it clearly and accurately, in the right context, as a part of the big picture. Yet the typical perspective view is a highly *distorted* view, as the following poem by Robert Graves points out.

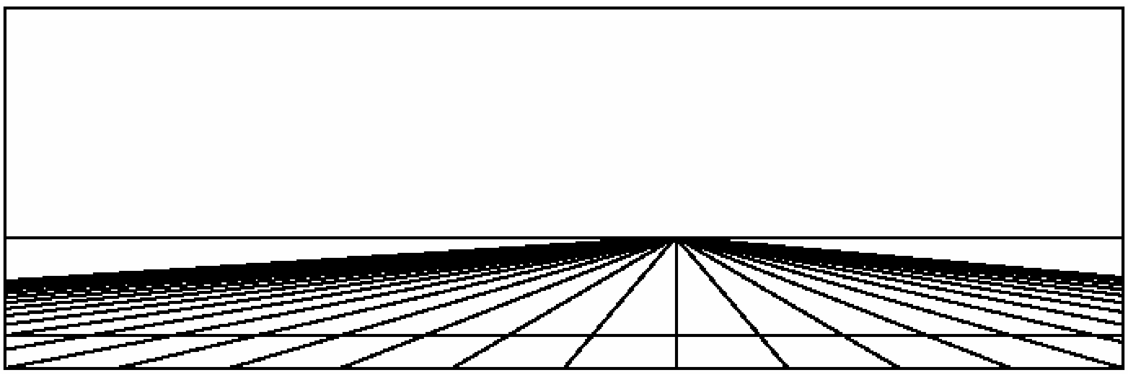
In Perspective

What, keep love in *perspective*—that old lie
Forced on the Imagination by the Eye
Which, mechanistically controlled, will tell
How rarely table sides run parallel;
How distance shortens us; how wheels are found
Oval in shape far oftener than round;
How every ceiling corner's out of joint;
How the broad highway tapers to a point—
Can all this fool us lovers? Not for long:
Even the blind will sense that something's wrong.

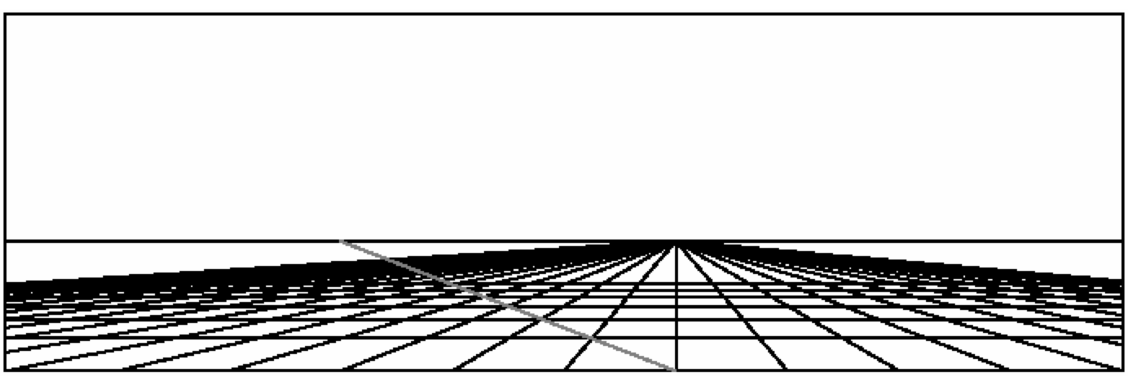
I suspect that we admire perspective views because it is not obvious how to draw 3-dimensional scenes correctly and artists, until comparatively modern times, didn't have a clue how to do it. True perspective introduces distortions—yes—but they are distortions seen by the eye. False perspective just looks laughably wrong. The following figure shows a naive medieval attempt to draw a 3-dimensional scene, alongside an engraving by Albrecht Dürer, made a few decades after the discovery of perspective drawing.

The method used by Dürer was first explained in a book on painting by the Florentine architect Leon Battista Alberti in 1436. It is still known under its Italian name, the *costruzione legittima* (legitimate construction).

The acid test of perspective drawing is to depict a tiled floor correctly. The picture on the left above fails this test miserably by drawing all the tiles "faithfully" with parallel sides, which makes the floor look vertical. The *costruzione legittima* takes a line of tile edges to coincide with the bottom edge of the picture and chooses any horizontal line as the horizon. Then lines drawn from equally spaced points on the bottom edge to a point on the horizon depict the parallel lines of tiles perpendicular to the bottom edge. Another horizontal line, near the bottom edge, completes the first row of tiles.



The tricky part comes next. How do we find the correct horizontal lines to depict the second, third, fourth, . . . rows of tiles? The answer is surprisingly simple: draw the diagonal of any tile in the bottom row (the line shown in grey below). The diagonal necessarily crosses successive parallels at the corners of tiles in the second, third, fourth, . . ., rows, hence these rows can be constructed by drawing horizontal lines at the successive crossings. Voilà!



The *costruzione legittima* succeeds by taking seriously what is "forced on the Imagination by the Eye": *that parallel lines meet on the horizon*. In the 17th century, the mathematicians Kepler (also famous as an astronomer), Desargues and Pascal developed this idea into a new kind of geometry, a geometry of vision called *projective geometry*. In projective geometry, the horizon is called the "line at infinity" and the point where a pair of parallel lines meet is called their "point at infinity".

Projective geometry does not conflict with ordinary geometry, but rather completes it by filling in the extra points that the eye expects. This is important for modern applications of geometry such as computer graphics. In designing video games, for example, geometry is one of the tools of the trade.

A Teaching Fellowship Project

As you now, every year a number of Teaching Fellowships are made available in a programme administered by the Royal Society. Maybe the project below is one that you might like to be involved in.

Statistics New Zealand, is the national statistical agency that is responsible for providing relevant and timely information on key aspects of New Zealand's economy, environment and society. We are looking for a primary and a secondary teacher who would be interested in investigating official statistics and how they are collected and used. They might also look at ways that statistics might be incorporated into the New Zealand curriculum. The projects would involve looking at the relevance of the Census of Population and Dwellings in New Zealand's society and see how this might be incorporated into learning in the classroom.

If you are interested in applying for a Fellowship for next year and you would like to work in the area above, then please contact:

Lesley Hooper,
email: lesley.hooper@stats.govt.nz,
ph 04 931 4924,
fax 04 931 4880.

Habits Of Mind by Andy Begg

Introduction

The title for this article comes from a paper that I recently read called *Habits of mind: an organizing principle for mathematics curriculum* (Cuoco, Goldenburg, & Mark, 1995). It focused my mind on a question that has been concerning me for some time. The question is, 'What do we want mathematics learners to be able to do?' This is a particularly important question that we need to be asking ourselves at the moment because people are talking about curriculum change. Traditionally the question may have been answered with, 'We want them to know the important mathematics listed in the curriculum'. More recently the answer may have been 'We want them to know and to use the mathematics they have learnt'. However, is knowing and using enough? I think not.

I also found this paper interesting because I could generalize it beyond mathematics and this generalization seemed to me to make good sense. When I am asked 'Why do we teach mathematics?' I usually reply 'We teach mathematics for the same reason that we teach all school subjects—because each different subject provides another way of making sense of one's world." But making sense of one's world in a mathematical way implies that one learns to think somewhat differently in mathematics and in each different subject. So, in saying knowing and using is not enough, I would suggest that we want learners to be able to and have the disposition to *think* mathematically as well as in many other ways.

When one looks at the aims of teaching a subject such as mathematics, for example on page 8 of Mathematics in the New Zealand Curriculum (Ministry of Education, 1992), one usually sees the normal sort of subject aims, but rarely is there a focus on 'making sense of one's world' from different perspectives, or a focus on mathematical thinking. For me making sense of one's world deserves more thought. I assume that one's world includes both one's work world (be it as teacher, learner, or whatever) and one's everyday world (the personal and social). One of the challenges that I see for teachers of our subject is to think about 'thinking', about the 'habits of mind' we want our students to develop, and then an even greater challenge, to think how these habits might be developed.

Now, what does it mean to think mathematically? I assume that all teachers of mathematics have implicit ideas about what they value in their students' thinking, but many have never made these ideas explicit. In that situation learners are unlikely to be sure what is being sought, and teachers are unlikely to ensure that thinking is assessed (which is one way of indicating that it is regarded as important). Perhaps therefore we should pause now, and you might jot down the ideas that come to mind under the heading 'mathematical thinking'.

…

PAUSE

…

From the curriculum you might have some key words like logical reasoning, problem solving, and communicating, but the curriculum documents for science and some other subjects would also have those words so one needs to think of these from a mathematical perspective.

Habits of mind

Now here is where I was excited by *Habits of mind*. The authors suggested that we are trying to have our learners develop *habits of mind* and these seemed to me be the *ways of thinking* that we would like learners to develop. The headings in their paper give some idea of what they see as the *habits of mind* for mathematics and as the paper can be accessed easily on the web I would suggest that you read it. The list of habits is in the table below but the paper gives a little more meaning to each of them. Note that the paper does not include the habits of mind of statisticians and as teachers in New Zealand schools it would be useful to fill this gap before thinking about curriculum.

Students should be pattern sniffers Students should be experimenters

Students should be describers Students should be tinkerers Students should be inventors Students should be visualizers Students should be conjecturers Students should be guessers
Mathematicians talk big and think small Mathematicians talk small and think big Mathematicians use functions Mathematicians use multiple points of view Mathematicians mix deduction and experiment Mathematicians push the language Mathematicians use intellectual chants
Geometers use proportional reasoning Geometers use several languages at once Geometers use one language for everything Geometers love systems Geometers worry about things that change Geometers worry about things that don't change Geometers love shapes
Algebraists like a good calculation Algebraists use abstraction Algebraists like algorithms Algebraists break things into parts Algebraists extend things Algebraists represent things

Table 1: Headings from *Habits of mind: an organizing principle for mathematics curriculum.* (From: Cuoco, Goldenburg, & Mark, 1995)

While the headings in the table suggest many desirable habits of mind it is possible that it is overwhelming and perhaps a clearer focus might provide something that is easier to focus on. Our curriculum hints at one such focus with the three processes—reasoning, problem solving, and communicating— but these have often been interpreted mainly within mathematics and not in terms of one's world in general.

Generalizing and conjecturing

Another focused perspective was provided for me in the last two years while I had the pleasure of working with John Mason at the Open University. He saw the most important 'habits' in mathematics as generalizing and conjecturing. He summed up how he saw these as important when he said that "Lessons that are not imbued with generalization and conjecturing are not mathematics lessons, whatever the title claims them to be" (Mason, 1996).

This focus on generalizing and conjecturing which is at the heart of both reasoning and problem solving has much to commend it. It seems to incorporate ideas such as of applying mathematics and of transferring beyond the immediate. It is comparatively easy to build into every lesson if one keeps it as a focus in planning. It is not difficult to think of examples; table 2 provides some as a starting point.

Geometry	With Pythagoras do we invite conjectures about $\angle A > 90°$ or $< 90°$ instead of $= 90°$? When do we explore the cosine rule as a generalization of the theorem. When do we think about shapes other than squares on the sides of the triangle
Number	When we teach operations do we generalize from whole numbers to negative numbers, decimals and fractions, and later to surds, and complex numbers?
Measurement	Do we generalize from using a ruler when we teaching protractor use?
Algebra	Do we always start with factorizing, multiplying and fractional work with numbers before we teach similar procedures with algebraic expressions?

Table 2: Generalizing and conjecturing

This notion of generalizing and conjecturing is worth pondering. When was the last lesson you taught where the learners were encouraged to conjecture and generalize? When I have asked teachers this some, mainly those who teach younger children, have said that it really only applies to more advanced mathematics, but I am sure that John Mason would say, and I would agree with him, "No, it applies at absolutely all levels from primary school through to the end of post-graduate work, and it applies in number, algebra, geometry, statistics, and all other mathematical topics."

Now, what 'habits of mind' would you like a new curriculum to focus on?

References

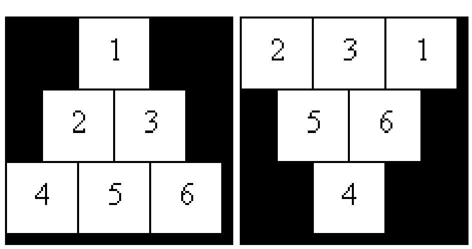
Cuoco, Al; Goldenburg, E Paul & Mark, June (1995) *Habits of mind: an organizing principle for mathematics curriculum*. http://www.edc.org/MLT/ConnGeo/HOM.html (Viewed on 16 Apr 2004) Later published in *Journal of Mathematical Behavior* 5(4) 375–402.

Mason, John (1996) Expressing generality and roots of algebra, In: Bednarz N, Kieran C & Lee, L (Eds.) *Approaches to Algebra,* Dordrecht: Kluwer Academic Publishers, pp. 65–86.

Ministry of Education (1992) *Mathematics in the New Zealand Curriculum*, Wellington: Ministry of Education.

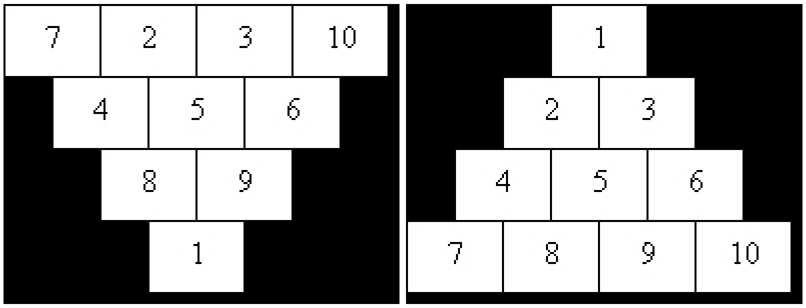
The Coins Problem By Cathy Walker

I spent some time working the coin problem from last month's newsletter. Now the first thing I discovered was that you can actually change the direction by moving 2 coins and not 3 as was said in the newsletter. For instance, you can change the coins direction using 1 and 4 (as in the diagram) or 1 and 6 or 4 and 6.

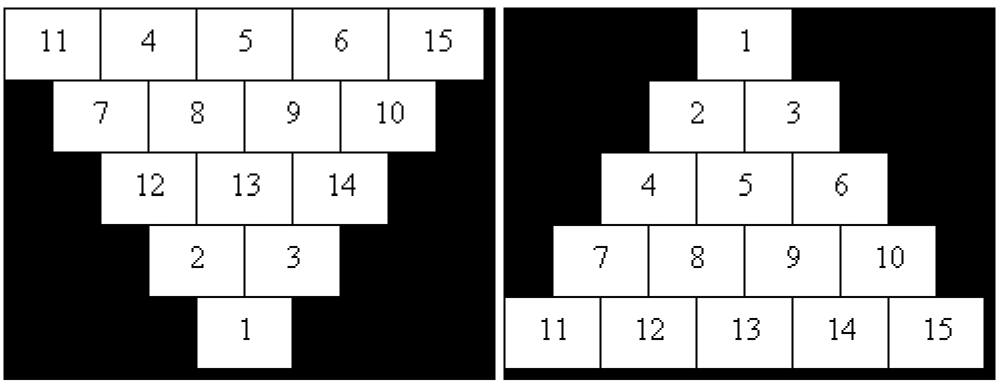


Changing direction by moving 1 and 4

With 10 coins I thought it might be 3 moves and sure enough moving 1,7,10 did the trick (see below).



I investigated further with the next triangular number 15 and found it took 5 moves - 1,2,3,11,15



This sounds like its heading toward a series – the Fibonacci series I think 1,2,3,5,8,13...... or not?

[We have to admit to our error. The question is does this problem generate the Fibonacci series? We'd appreciate your input for the next newsletter.]

Solution to April's problem

The problem was to find the largest number of pigeonholes in which 1000 doves could be allocated, each hole containing a different number of birds.

Since the largest number of holes is sought, the birds must be allocated in the smallest different numbers possible, i.e. 1 in the first hole, 2 in the second and so on. The problem thus revolves round the sum of the natural numbers 1 + 2 + 3 + ….. , finding when they first sum to 1000.

One can approach this with some knowledge of arithmetic sequences. In this example, with first term and common difference both 1. A good place to start is to find how many terms are needed for the sum to be 1000.

Trial and error may be an alternative route. Either way you will soon discover that
$1 + 2 + 3 + ... + 44 = 990$ and $1 + 2 + 3 + ... + 45 = 1035$.

Hence 44 is the largest number of pigeonholes that can house any number of doves from 990 to 1035 with no two holes containing the same number of doves. The answer to the problem is therefore 44.

We had several solutions this month so the competition was the highest it's been for a while. We thought that the best solution came from matt walker

Congratulations Matt. The voucher is on its way.

For some more information on arithmetic progressions you might like to look at The Why and How of General Terms, Algebra, level 6.

This Month's Problem

Using each of the ten digits once, find the two five-digit numbers which have the largest possible product.

We will give a petrol voucher to one of the correct entries. Please send your solutions to derek@nzmaths.co.nz and remember to include a postal address so we can send you the voucher if you are the winner.

Afterthoughts

Other ideas you could use for a balloon debate (assuming five in the balloon);

(1) x^2, x^3, 2x, 3x, 4x
(2) any five numbers, say: π, 5, 5.5, 7.26, 10
(3) length, width, circumference, perimeter, radius
(4) centimetre, degree, kilogram, newton, second
(5) complementary angles, corresponding angles, alternate angles, supplementary angles, vertically opposite angles
(6) prime number, square number, even number, cube number, triangular number
(7) Pythagoras, Euclid, Newton, Leibniz, Alkarismi

Some of these choices offer opportunities for research. They make a good homework project.