

Newsletter No. 31
March 2004
Welcome to the second newsletter of 2004. We have lots of goodies for you this month so let's get under way.

Maths is commonly said to be useful. The variety of its uses is wide but how many times as teachers have we heard students exclaim, "What use will this be when I leave school?" I guess it's all a matter of perspective. A teacher might say mathematics is useful because it provides him/her with a livelihood. A scientist would probably say it's the language of science and an engineer might use it for calculations necessary to build bridges. What about the rest of us?

A number of surveys have shown that the majority of us only need to handle whole numbers in counting, simple addition and subtraction and decimals as they relate to money and domestic measurement. We are adept in avoiding arithmetic - calculators in their various forms can handle that. We prefer to accept so-called ball-park figures rather than make useful estimates in day-to-day dealings and computer software combined with trial-and-error takes care of any design skills we might need. At the same time we know that in our technological world numeracy and computer literacy are vital. Research mathematicians can push boundaries into the esoteric, some of it will be found useful, but we can't leave mathematical expertise to a smaller and smaller proportion of the population, no matter how much our students complain.

Approaching mathematics through problem solving - real and abstract - is the philosophy of the nzmaths website. It stimulates, it involves and it works.
"Mathematics is most important madam! I don't want to have you like our silly ladies. Get used to it and you'll like it. It will drive all that nonsense out of your head."

> Tolstoy (War and Peace)

## What's new on the nzmaths site this month?

The biggest development on the nzmaths site this month is the launch of the Online Numeracy Workshops. Unfortunately, this is the only area of the site that is not freely available online. This is because the workshops are password protected and require videos that are only available on CD-ROM. If your school has completed the Numeracy

Project professional development, then you can expect to receive a copy of the CDROM required to run the workshops in the near future. If your school has not completed the professional development or you want additional copies (or you are not attached to a school) then you should be able to buy them from resource centres at Colleges of Education.

## Diary Dates

A couple of dates in the second half of the year that are worth thinking about now:
Maths Week is the $9^{\text {th }}$ to the $13^{\text {th }}$ of August this year, but now may be the time for the more organised teachers among us to start thinking about how to work it into their long term plans. The website will be up and running from the start of term 3 for students to start earning credit towards Maths Week auctions.

If the item in last month's newsletter has inspired you to apply for a 2005 Teaching Fellowship you have until the $16^{\text {th }}$ of July to apply. We encourage interested teachers to look at the application form sooner rather than later as it is very comprehensive.
Information and application forms can be found at http://www.rsnz.org/awards/teacher fellowships/ .

## Perfect Numbers and their Offspring

In the third century BCE Euclid defined the so-called perfect numbers as those numbers that are the sum of all their divisors apart from themselves. These divisors are called the aliquot parts. For example, 6 is perfect since $6=1+2+3$, and $28=1+2+4$ $+7+14.6$ and 28 are the two smallest perfect numbers. The next two, 496 and 8128, were the only others that Euclid knew. It was not until the Middle Ages that the next highest perfect number 33550336 was found.

The only perfect numbers known are of the form $2^{n-1}\left(2^{n}-1\right)$, where the second factor is prime. It is not known how many perfect numbers exist but as we increase from one to the next they certainly thin out very quickly. They could disappear altogether, or there could be heaps more among all those unimaginably large numbers. The largest perfect number, as far as I know, was found in December 2001 with the aid of a computer. It has four million digits. It was the $39^{\text {th }}$ to be found but not necessarily the $39^{\text {th }}$ perfect number as there may be smaller undiscovered ones.

More recently, the idea of perfect numbers has spawned a whole family of others;
Almost perfect numbers are those with their aliquot parts summing to one less than themselves. For example, 8 is an almost perfect number since $1+2+4=7$. All powers of two are almost perfect numbers. It is not known whether an odd almost perfect number exists.

Quasi-perfect numbers have their aliquot parts summing to one more than themselves. Although no quasi-perfect numbers have been found any that does exist must be the square of an odd number.

Semi-perfect numbers are those which are the sum of some of their aliquot parts. The first three semi-perfect numbers are 12, 20 and 24. Can you find the fourth? [The answer is below.]

Multiply perfect numbers have their aliquot parts summing to multiples of themselves. 672 is a multiply perfect number since its aliquot parts sum to $1344=2 \times 672$. All perfect numbers are by definition also multiply perfect. Can you find the smallest multiply perfect number that is not perfect? [The answer is below.]

Abundant numbers are numbers that are less than the sum of their aliquot parts. The first three are 12, 18 and 20. Can you find the fourth? [The answer is below.]

Weird numbers are those which are abundant but not semi-perfect. That is, they are less than the sum of their aliquot parts but not the sum of any set of them. They are rare. 836 is one but there is another much smaller. Can you find it? [The answer is below.]

Deficient numbers are those that are more than the sum of their aliquot parts. Most numbers are deficient.

Amicable numbers are pairs of numbers which are each the sum of the aliquot parts of the other. The smallest such pair is 220 and 284. Pythagoras is said to have known of this pair but no others. Perhaps that is not surprising as the next smallest pair is 17296 and 18416 .

Sociable numbers are sets of three or more numbers each with their aliquot parts summing to a different one of the others. That is, the aliquot parts of the first sum to the second, the second's aliquot parts sum to the third and so on, with the sum of the aliquot parts of the last number equalling the first. Sociable numbers thus form a cycle. The smallest two sociable numbers have cycles of 5 and 28 . There are some sociable numbers with cycles of four but none have been found with three. The smallest sociable numbers are $12496,14288,15472,14536$ and 14264.

Untouchable numbers are never the sum of the aliquot parts of any other number. The two smallest untouchable numbers are 2 and 4 . What is the next smallest? [The answer is below.]

## Polar Nonsense

You've probably all heard the problem about the hunter who travels 3 km South, and then 3 km East, at which point he shoots a bear. He then goes 3 km North and gets back where he started. The problem is what kind of bear was it?

The traditional answer is that it is a polar bear because the only way that the hunter could do what he has supposedly done is for him to have started and ended at the North Pole.

When you think about it, there are two places on Earth where the parallel of latitude is 3 km in circumference. The one in the Southern Hemisphere the hunter could have got onto by going 3 km South from a number of places. In going 3 km East, he would then go all round the Earth so that his trek 3 km North would take him back to his starting point. Now I suppose that there are not usually any bears at all that close to the South Pole, so perhaps the hunter wasn't there after all but we just thought you should know that the possibility had to be considered.

Now, of course, at the South Pole, no matter what direction you face your compass faces North. But what happens to the needle at the North Pole. Does it go berserk?

And the other problem down or up there is the time. At the Pole what is the time? Do they take Greenwich Mean Time? Even if they do, just a few metres away from the Poles you can pass through all the time zones and cross the Date Line in no time flat.

What else is weird there?
In fact, where else is weird? How about the equator? It turns out that water won't go down the plug hole if the hole is on the Equator. You know that in the Northern Hemisphere water goes down round one way and in the Southern Hemisphere it goes round the other way. If you get a hole on the Equator the water's not sure what to do so it stays where it is.

## Solution to February's Problem

First of all let's tell you the official story. And then we'll confess to all. First the problem and then our solution.

When sending a birthday card to her father, a day that coincided with her own birthday, Samantha realised that their ages would both belong to the select set of numbers which could not be expressed as the sum of consecutive integers. So how old will each of them be on their birthday?

The solution to last February's problem depends on knowing about numbers that cannot be expressed as the sum of consecutive integers. Let's do a bit of investigating, For example;

$$
\text { and } \quad \begin{aligned}
2+3+4+5 & =14 \\
& 1+2+3+4+5=
\end{aligned}
$$

but no set of consecutive integers can be found to sum to 16 .

This is an investigation that would benefit from being carried out as a group activity where contributions from different individuals would give new insights to the problem.

It shouldn't take long for someone to see that any odd number, other than 1 of course, can be expressed as the sum of two consecutive integers, for example, $3=1+2,5=2+3,7=3+4$ and in general, $2 \mathrm{n}+1=\mathrm{n}+(\mathrm{n}+1)$.

Other patterns emerge, such as the sum of four consecutive numbers will always give an even number as it will always contain two even and two odd numbers. It will also be seen that the solution to a given number is not always unique. 15 , for example, can be expressed as $7+8,4+5+6$ and $1+2+3+4+5$.

Having quickly dismissed the odd numbers the problem resolves itself into finding which of the even numbers can be represented. Consider the sum of three consecutive numbers, for example, $10+11+12$.
They can be thought of as $(11-1)+11+(11+1)=3 \times 11$. In general the sum of three consecutive numbers will always be a multiple of 3 since

$$
(\mathrm{n}-1)+\mathrm{n}+(\mathrm{n}+1)=3 \mathrm{n} .
$$

Further, this shows that all multiples of 3 can be expressed as the sum of three consecutive integers.

Consider now the sum of five, seven, nine, $\ldots,(2 n+1)$ consecutive integers in the same way.

Five numbers: $(\mathrm{n}-2)+(\mathrm{n}-1)+\mathrm{n}+(\mathrm{n}+1)+(\mathrm{n}+2)=5 \mathrm{n}$ Seven: $(\mathrm{n}-3)+(\mathrm{n}-2)+(\mathrm{n}-1)+\mathrm{n}+(\mathrm{n}+1)+(\mathrm{n}+2)+(\mathrm{n}+3)=7 \mathrm{n}$
and in general the sum of p (odd) consecutive numbers will be pn , a multiple of p where n is the middle number of the sequence.

Summarising the above, we have shown that any number with an odd factor can be expressed as the sum of consecutive numbers. This leaves only numbers with no odd factors, i.e. the powers of 2. Can they be expressed as the sum of an even number of consecutive integers?

Four numbers: $(\mathrm{n}-1)+\mathrm{n}+(\mathrm{n}+1)+(\mathrm{n}+2)=4 \mathrm{n}+2=2(2 \mathrm{n}+1)$
Six: $(n-2)+(n-1)+n+(n+1)+(n+2)+(n+3)=6 n+3=3(2 n+1)$
In general, the sum of 2 q consecutive numbers is $\mathrm{q}(2 \mathrm{n}+1)$. From this it is clear that an even number of consecutive integers always sums to a number that has an odd factor, so cannot be a power of 2 .

We are thus left with the final conclusion that all numbers except powers of 2 can be expressed as the sum of a set of consecutive integers.

That only leaves us to answer the original problem. Samantha is 32 and her father 64 (although 16 and 32 are possible if her father was a precocious 16 years old!)

Then along came Cassandra Li a Year 9 student at Columba College. She pointed out that there are no integers that cannot be expressed as the sum of a string of consecutive integers. The point is that integers can be negative. So, for example,

$$
4=-3+-2+-1+0+1+2+3+4
$$

In fact 4 , and any other integer for that matter, can be written in an infinite number of ways as the sum of consecutive integers.

Cassandra then rightly pointed out that there was no unique answer, as we had implied in the question. Of course we noted the difficulty in our solution but maybe that isn't quite good enough.

Now Cassandra wasn't the only one to point out both of our sins. We had an email from Trident High School in Whakatane to tell us the same thing. But the reason that Cassandra gets the voucher this month is that she also pointed out that we gave an incorrect solution to the November problem. Apparently in that question there was a twin but in our solution we gave all different ages. We'll look into that difficulty. That may be why we didn't have a winner last month.

## This Month's Problem

We're going for something completely different this month - pure creativity. Back in the 1980s recreational mathematician Reg Alteo invented what he called 'Twoons', where the number 2 and various mathematical contrivances were used in the writing of song titles. For example:

On a bicycle made for $\mathbf{2}^{1}=$ 'On a bicycle made for two' You were made $\mathbf{2}^{2} \mathbf{m e}=$ 'You were made for me' $\mathbf{2}^{1.585}$ (approximately) coins in a fountain = 'Three coins in a fountain' Life begins at $\mathbf{2}^{2} \times \mathbf{1 0}=$ 'Life begins at forty' When I'm $\mathbf{2}^{\mathbf{6}}=\mathbf{}^{\mathbf{\prime}}$ When I'm Sixty-four'.

What we're looking for this month are more 'twoons', or 'throons' perhaps, even 'noons'. The idea is to encourage the more divergent thinkers among you. Feel free to interpret the task as you wish.

We will give a petrol voucher for the entry that we think is the most exhaustive and/or creative. Please send your solutions to derek@nzmaths.co.nz and remember to include a postal address so we can send the voucher to you if you are the winner.

## Answers to Perfect Numbers and their Offspring from above:

1. The fourth semi-perfect number is 30 .
2. The smallest multiply perfect number that is not perfect is $\mathbf{1 2 0}$.
3. The fourth abundant number is 24 .
4. The smallest weird number is 70 .
5. The next smallest untouchable number is 52 .

## Afterthought

I heard this delightfully ambiguous advert on the TV recently:
"Join Jenny Craig now and get 50\% off"

It suggested to me the idea that we might try and collect such items. How about sending us any that you find. To get started here's another that I spotted outside our local coffee shop:
"All food made fresh on the premise"

On the premise of what, one wonders!
We will surely find a book voucher for the best entries we receive.
In the same vein, there's at least one ornithologist amongst us and he couldn't resist taking this picture.


The sign is on a main road just north of Mackay, Queensland. It's near a local wetland inhabited by Brolgas. These large birds are in the Crane family and it takes them a bit of effort to get above car height.

Have a good month.

