

Newsletter No. 17
September 2002
The nzmaths website has as one of its basic philosophies that maths can be better understood when approached through a bit of problem solving. Indeed, Derek Holton once said, "Problem solving encapsulates what mathematics is all about." Perhaps our own preoccupations with mathematics centre round problem solving at school level but l'm sure it doesn't come as a surprise to hear that problem solving is what research mathematicians do. Indeed, with the recent successes of films like A Beautiful Mind and Good Will Hunting and books like Simon Singh's Fermat's Last Theorem, mathematical research and by implication problem-solving, is becoming quite the glamour event. This, of course, can only be good for mathematics.

Mathematicians are happy to solve problems for the glory of it alone but there have been over the years some financial rewards available for solutions to certain 'sticky' problems.
Prizes, of course, are routinely awarded in magazines for solutions to 'this month's problem', even this newsletter is not exempt. The Hungarian mathematician Paul Erdös set hundreds of problems in his published papers and via letters to colleagues with prizes starting at 25 cents, totalling around $\$ 25,000$. Some of mathematician's more complicated problems that have stood the test of time carry huge rewards. Martin Dunwoody is hoping to collect a million dollars for his proposed solution to a centuries-old problem set by Poincaré and there is a million dollars offered for each of the solutions to a number of others including the Reimann hypothesis and the Hodge conjecture.

Not all of the problems are 'oldies'. Research mathematicians Alex Selby and Oliver Riordan shared a million dollars in October 2000 for solving the Eternity Puzzle, a 209-piece 'jigsaw from hell'.

Where does that leave us? Well, I guess you can have a go at this month's problem in our newsletter. I'm not sure Derek will offer a million dollars for the solution but you might pick up fifty.

## In Mathematics the art of posing problems is easier than that of solving them.

Georg Cantor

If you'd like to know more about the million dollar problems - what they are and how to go about collecting the prize money for their solution, you might like to look at
www.claymath.org
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- What's new on nzmaths.co.nz
- Minus times minus equal plus
- More on the problem solutions for June and July
- Problem of the month


## WHATS NEW ON THE NZMATHS SITE THIS MONTH?

Another ten websites links have been added to the links component of the site but apart from them nothing new has been added this month.

## Minus times minus equals plus

It just came up in conversation the other day, as it often does. There are these mantras in maths that we all parrot off, and use if we have to, but we don't really know why they are true. Come on now. If one of your students asked you to say why it's true what would you say? Why does 'minus times minus equal plus' work?

So what did we decide? Well, we came up with two arguments. One was more a demonstration than anything else: what else could it be? The other was a tight, in fact watertight, argument based on sound algebraic principles. The former was something that you could explain to anyone. The latter, well ... So here's a distillation of the arguments. See what you think. Are you convinced?

Try 1: OK so let's look at ( -3 ) x ( -4 ). What could this possibly be? Let's sneak up on that. First we all know that $3 \times 4=12$. What about $(-3) \times 4$ ? Surely this is 4 lots of $(-3)$ ? And that has to be $(-3)+(-3)+(-3)+(-3)=(-12)=-12$. (We're just using the brackets here so that you're not tempted to do any operating with them.)

You can probably do the calculation many other ways. Think about (-3) as being your overdraft in millions of dollars. Right? Then (-3) 4 is essentially 4 lots of such redness. That has to be an overdraft of 12 million dollars. So $(-3) \times 4=-12$.

That brings us to the more important question of the value of $(-3) \times(-4)$. It's hard to think up a 'real' situation for that so we have to get there through the backdoor. Step 1 says well there's a 3 there and a 4 so there ought to be a 12 in the answer. Step 2 says that, because of the minus signs, the answer will surely be +12 or -12 . But $(-3) \times 4=-12$, so most likely $(-3) \times(-4)=12$.

We can tighten this up a bit if we look at $[(-3) \times 4]+[(-3) \times(-4)]$. What is this? Now
$[(-3) \times 4]+[(-3) \times(-4)]=(-3)[4+(-4)]=(-3)[4-4]=(-3) \times 0=0$. Whatever $(-3) \times$ $(-4)$ is then, when you add it to $(-3) \times 4$ you get zero. But $(-3) \times 4=-12$. What do you have to add to -12 to get zero? Surely that's +12 . So $(-3) \times(-4)$ has to be $+12!$ !

Now what you can do for (-3) x (-4) you can do for any two negative numbers. So that has to mean that $(-a) \times(-b)=+a b$. Minus times minus does equal plus!!

Try 2: We have to say at the start that this is not for the faint hearted. We also have to say that all we are about to do is to use the method of Try 1 but use algebra and so gie a justification for all a and b not just 3 and 4 .. So what is $(-\mathrm{a}) \mathrm{x}$ (-b)?

First we need to convince you of something you were probably happy about anyway but convince you we will. (-a) $x \mathrm{~b}=-\mathrm{ab}$. How do we do this? Consider (a) $x b+a \times b$. So
$(-a) \times b+a \times b=[(-a)+a] \times b=[-a+a] \times b=0 \times b=0$.
If $(-a) \times b+a \times b=0$, then $[(-a) \times b]=-[a \times b]=-a b$. And that's just what we were trying to show: $(-a) \times b=-a b$. In pretty well exactly the same way we can show that $\mathrm{a} \times(-\mathrm{b})$ also equals -ab .

But what about $(-a) \times(-b)$ ? First notice that $[(-a) \times b]+[(-a) \times(-b)]=(-a)[b+(-b)]$ $=0$. So $[(-a) \times b]=$ which gives $-[(-a) \times(-b)]=-a b$, so $[(-a) \times(-b)]=a b$. And minus times minus does equal plus!!

If that doesn't make any sense to you then email Derek at derek@nzmaths.co.nz . He's got nothing much else to do and he'd be glad to explain to you the bits that you didn't understand. (But first try the above argument with a few specific values of $a$ and $b$.)

## More on the problem solutions for J une and J uly

We've had a few more thoughts on the solutions for June and July. So we've put them below. We also realize that the August newsletter was emailed to subscribers but it did not reach the web site until very late in the month. So we are extending the August problems for one more month. Sorry about those of you who have emailed in your answers already.

## 1. June Solutions

Recall that we had three discs (below) with numbers on both sides. We tossed the discs in the air, noted the numbers that were face up, and added the three
numbers together. Then we saw that the totals were always consecutive numbers. So what numbers were on the other side?


One way to tackle this problem is to throw one disc away. Try to see what happens in an easier case and build up to the harder one. In fact it would be even easier if we just had discs with 0 and 0 on the top. With 0 and 0 we'd have to make totals of $0,1,2$ and $3 ;-1,0,1$ and $2 ;-2,-1,0$ and 1 ; and $-3,-2,-1$ and 0 .

So what numbers would we have to use on the other side of the zeros to get these consecutive totals? 1 and 2 would get the first string; -1 and 2 the second; 2 and 1 the third and -2 and -1 the fourth. So now add 5 and 8 to the 0 sides and add $/$ subtract the 1 s , and 2 s to 5 and 8 and put them on the other side of the disc. But watch out. $5+1$ and $8+2$ as well as $5+2$ and $8+1$ both give consecutive totals of $13,14,15,16$. So it looks as if here there are eight ways to get the four consecutive totals.

Now for those of you who have not forgotten all of the algebra that you were taught, here is a start to an algebraic approach to the discs problem. Suppose that the number $a$ is on the back of the disc with 5 showing and the number $b$ is on the back of the 8 . One way to get consecutive numbers is for $5+8=13 ; a+8$ $=14 ; 5+b=15$ and $a+b=16$. If you solve that you get $a=6$ and $b=10$. (Now all you have to do is to write out the other possible equations!)

But have you ever come across the following problem. We think that we saw it somewhere in the Figure It Out series. On his old fashioned pan scales, a butcher can weigh out each of $1,2,3,4,5,6,7,8,9,10,11,12,13,14$, and 15 kg exactly. How can he do this with only three weights?

We suppose that we had better get back to the three discs. Now the same thing happens with three discs as it did with two. But this time we have not just to add or subtract 1 and 2 but add or subtract 1,2 and 4 . Discs that have 0 with 1,0 with 2,0 with 4 on them, give all totals from 0 to 7 . Adding these together with 5,8 and 11 give 6 possibilities. But if we think about $\pm 1, \pm 2$ and $\pm 4$ we get a whole lot more. There are 8 ways to assign the plus or minus to three numbers. So altogether we get 48 possible ways to add numbers to the discs.

The best crop of answers was from Cathy Walker. She produced 20. Here they are: $1,6,12 ; 1,7,9 ; 1,10,10 ; 3,7,7 ; 3,4,10 ; 3,9,7 ; 4,6,7 ; 4,4,9 ; 4,10,7 ; 6,6,7 ; 6,4,9 ;$ $6,10,7 ; 6,12,9 ; 7,7,7 ; 7,4,10 ; 7,9,7 ; 9,7,9 ; 9,6,10 ; 9,9,9$; and $9,10,10$. So she gets the book voucher for the answer to the June problem. But we have to give a special mention to the spreadsheet submission that tried to systematically produce all of the answers. WE couldn't get it to work but it is a way to go. It will certainly go through all possible numbers for you, picking up all the correct answers on the way

## 2. July Solutions

And talking of spreadsheets, one way to get your hands on the farmers and their sons is to use one of those new fangled inventions.

In column A put the numbers 1 up to n (where n was the number of calves, and hence the price per calf, that a son could have bought); make column B, column A squared (the amount the son paid); make column C, column B + 63 (the amount the son's father paid); and make column D the square root of column C (the number of calves, and price, the father bought).

No matter how far you extend the spreadsheet there are only three values of $n$ which produce a whole number for $\left(n^{\wedge} 2+63\right)^{\wedge} 0.5$, so these must be the father/son pairs. These numbers are 1, 9 and 31 . Given that Alan buys 23 calves more than Ernie and Craig buys 11 more than Dan the only possible combinations are:

Alan's son is Fred, Craig's son is Ernie, and Bob's son is Dan.
The only correct solution came from Evan Jones from Napier.

## This Month's problem

Imagine a square of paper. Can you cut it up into smaller squares? It certainly doesn't sound difficult. For example,

A square can be cut into .... four squares, or ..... six squares.


Nobody mentioned that the squares had to be the same size!

The question is, can we cut our square into any and every number of squares? That means, of course, with no paper left over.

Each month we give a petrol voucher to one of the correct entries. Please send your solutions to derek@nzmaths.co.nz and remember to include a postal address so we can send the voucher if you are the winner.

> Mathematics is not a spectator sport. If you want to be good at it you have to practise, practise, practise.

Dennis McCaughan

