Mathematics in the New Zealand Curriculum
Second Tier

Strand: Statistics — Thread: Probability — Level: 3

Achievement Objective:

- Investigate simple situations that involve elements of chance by comparing experimental results with expectations from models of all the outcomes, acknowledging that samples vary.

Exemplars of student performance:

Exemplar One: Tania recognises that in drawing one marble from a can containing two blue and three red marbles, a red marble occurs more often than a blue marble, even though the first ten trials produce variable results to the contrary.
She accepts short-term variation from a small number of trials (small sample) rather than questions her initial prediction. Tania recognises that greater sample size will result in a tighter fit between her prediction of more red marbles being drawn than blue and the results:

Red	Blue
𝍷𝍷𝍷𝍷	𝍸 𝍷

Initial sample of 10 trials

Red	Blue
𝍸 𝍸 𝍸 𝍷	𝍸 𝍸 𝍷𝍷𝍷𝍷

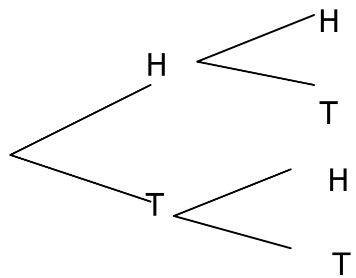
Increase sample of 30 trials

Tania's thinking exemplifies Level Three because she acknowledges all of the possible outcomes in believing that a red marble is more likely than a blue marble. She is able to accept that variation occurs with sampling through maintaining her prediction of more red marbles despite a sampling result to the contrary.

Exemplar Two: Deacon and Gabrielle play a game where two coins are tossed, one player winning if the coins turn up the same (two heads or two tails), the other winning if they are different (heads-tails). They recognise, after an increasing number of coin tosses, that the each player wins about half the time. They accept that the results are "around about" one half of the outcomes are heads and one half are tails. Deacon and Gabrielle develop a model of all the possible outcomes (e.g. list, tree diagram, and table), and compare their model to the results of trialing (coin tossing).

Coin A/Coin B	Heads	Tails
Heads	HH	HT
Tails	TH	TT

Deacon records a table to show HH and TT occur as often as HT and TH.



Gabrielle uses a tree diagram to show that 2 out of 4 possibilities were HT and TH.

The students recognise that their theoretical models match the results of their coin tossing. They accept that the game is fair and accept that both players have an equal chance of winning.

Deacon and Gabrielle's thinking exemplifies level three because they realise that even after a large sample of trials that there is still likely to be variation from the theoretical probability of 50-50. They accept that the results vary from exactly one half TT and HH and one half TH and HT. They construct appropriate representations (table and tree diagram) to work out the theoretical probabilities and connect their models to the results of coin tossing.

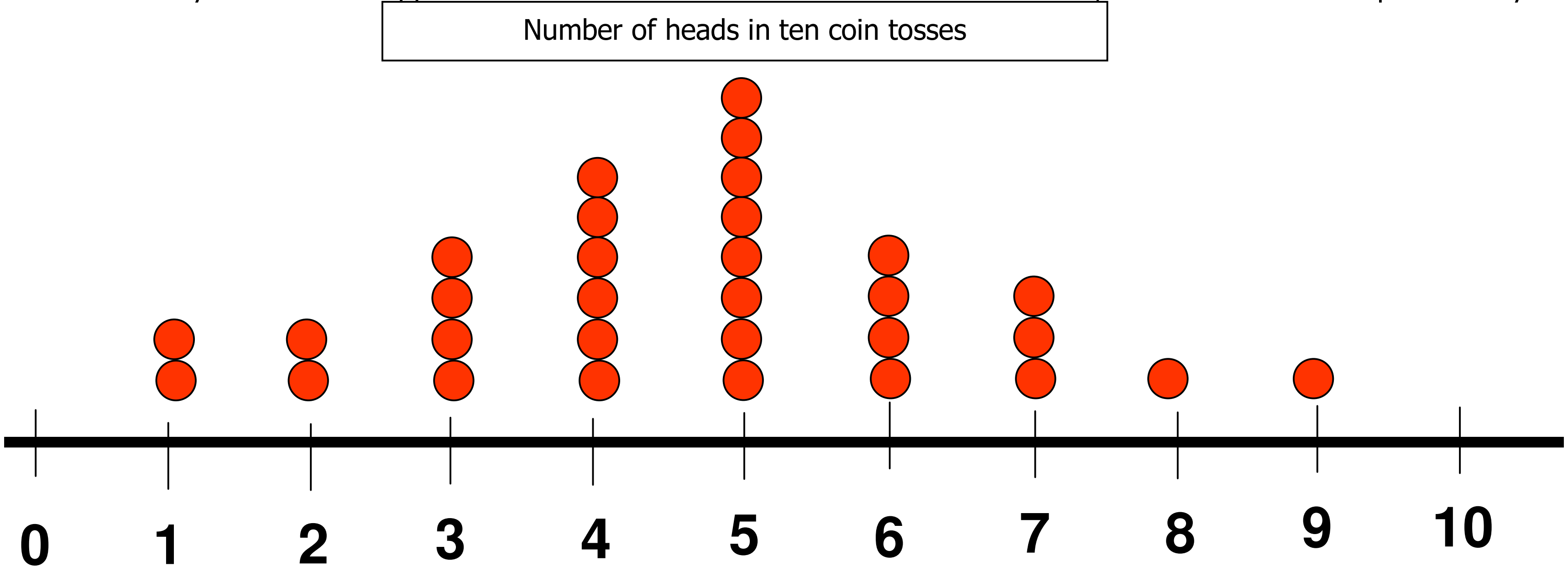
Exemplar Three: In a game of tossing two dice and finding the product, e.g. $4 \times 3 = 12$, Andre recognizes that scores of zero and over 36 are impossible. He explains that a score of zero would only be possible if there was a zero on one or both of the dice. He adds, "Since the largest number on both dice is six the highest possible total is $6 \times 6 = 36$". He also states that the product will always be in the range one to thirty-six. He is certain about this being always true.

Andre's thinking exemplifies level three because he realizes when outcomes are certain or impossible based on a model of all the possible outcomes.

Important teaching ideas:

The focus at level three is on helping students appreciate that natural variability occurs in samples (collections of events) of the same experiment or situation and that each event is independent of previous events, no matter how convincing a pattern of occurrence may look. For example, if a coin is tossed ten times the results might be 5 heads-5 tails or 4 heads-6 tails or 7 heads- 3 tails, etc. If all the students in the class carry out ten coin tosses there will be a range of results. The results are variable and this variation is natural. It is not a result of flawed coin tossing technique, biased coins or any other experimental feature.

The common expectation that 5 heads-5 tails is the most likely result comes from a mathematical model of all the possible outcomes. In reality this is what happens. While 5 heads-5 tails is the most common result, other results feature prominently.



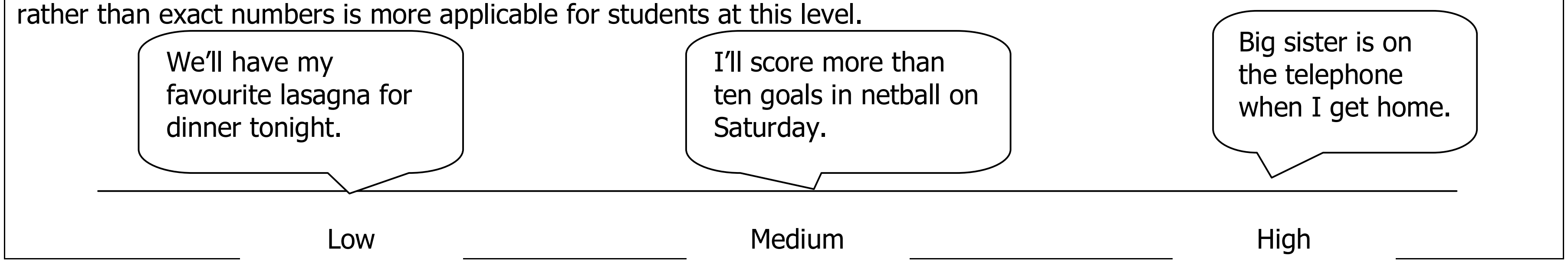
Independence of results is an important idea in probability. This means that a previous result does not influence the next one. Suppose you toss a coin ten times and get 2 heads - 8 tails. That does not mean that in the next ten tosses you are more likely to get tails than heads, to redress the balance, or less heads than tails because the coin is automatically biased. Each ten tosses is an independent event. There is no influence of the previous coin tosses to the next toss.

Provide the students with simple chance events where all of the outcomes are easy to find visually or by creating lists, tables or

diagrams, e.g. Using a bag of counters to model a lake full of fish (four red, three blue, two yellow, one green).
Get each student to do ten trails of taking a “fish” from the lake then replacing it. Ensure that the students record the sequence of “fish”, e.g. R B Y R R B Y R G R, as well as a tally of frequency (how many of each fish in total).
Ask the students to compare their results from the same event (fish taking) with a partner. They should notice variability in both the sequential and frequency results.
Ask them to predict the next “fish” and justify their prediction. Look for acceptance of both uncertainty and independence. Uncertainty refers to not knowing what is likely to occur, e.g.”It could be any of the colours.” Independence refers to the previous events not affecting the next, e.g. “The last five fish have been B R R R R but the next fish could still be red.”
Discuss with the students how reliable their predictions were. This should show that a prediction of red was marginally more reliable than blue. Ask why this occurred. This connects to the theoretical probability. Since the possible outcomes are R R R R B B B Y Y G, there was a four in ten chance of the next fish being red and a three in ten chance of it being blue. Neither theoretical probability was grounds for confidence, let alone certainty.
Key points are:

- Small samples vary a lot in both sequence and frequency
- What happened before has no effect on what happens next
- By thinking about all of the outcomes we can make uncertain (in most cases) predictions of what might happen

Ensure that students are exposed to events where the actual probabilities can only be approximated by trialing, like dropping drawing pins and finding the probability of them landing safely (point down). Most statistical investigations in the real world involves situations that cannot be predicted theoretically, e.g. What percentage of the population is left-handed? Researchers experiment or survey a part of the group they are interested in. The part is called a sample, the whole is called the population. From their results researchers estimate the probability, e.g. 125 out of 1000 people are left-handed so about 12.5% of the whole population is left-handed. Note that any predictions are uncertain but the larger the sample, the more reliable the predictions are (usually).
Connect the language of probability with simple fractions. For example, discuss different representations for a half chance, like “fifty-fifty”, “even chance” and “equally likely”. Represent predicted and trialed probabilities with simple fractions. Get students to think about everyday situations in which probability is involved and assign simple probabilities to them. A qualitative feel rather than exact numbers is more applicable for students at this level.



Broaden students’ methods for establishing all the outcomes by involving them in events of two stages. These events might be conditional, that is depending on one another, and some might be independent. For example, drawing two marbles out of a can containing two red marbles and three blues are conditional events as the marble selected first effects the chances for the second marble. Tossing two dice and finding the product involves two independent events, though the product depends on both outcomes.

Students should be encouraged to develop their own methods for recording all the possible outcomes of events. Gradually they can be exposed to more formal methods for the marbles in the can situation, such as:

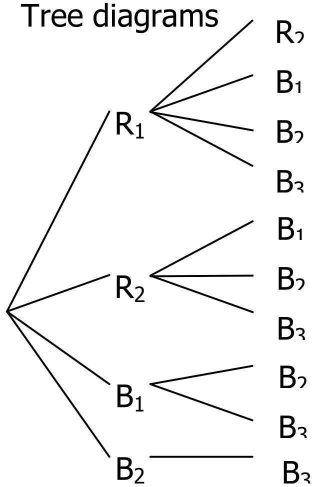
Organised lists

Red 1-Red 2
Red 1-Blue 1
Red 1-Blue 2
Red 1-Blue 3
Red 2-Blue 1
Red 2-Blue 2
Red 2-Blue 3
Blue 1-Blue 2
Blue 1-Blue 3
Blue 2-Blue 3

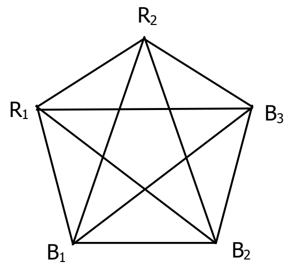
Tables

	R_1	R_2	B_1	B_2	B_3
R_1		☑	☑	☑	☑
R_2			☑	☑	☑
B_1				☑	☑
B_2					☑
B_3					

Tree diagrams



Networks

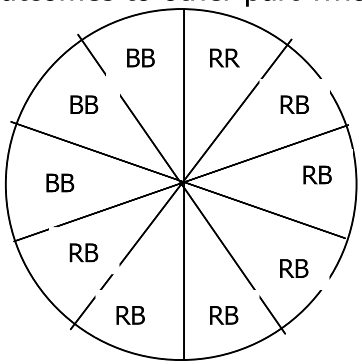


Build these methods of recording through trialing. For example, label the marbles R_1, R_2, B_1, B_2, B_3 and trial draws of two marbles, building up the tree diagram, or organised list, as different outcomes appear. As with simple one-stage events link the possible outcomes to long-term trialing of the situation concerned. Record the results as the sample size increases using tally charts or bar graphs.
Get students to reflect on the match between their theoretical prediction from all the outcomes and the results as the sample size increases. In general, the results get proportionally closer to the theoretical expectation as the sample size increases. This idea develops further at level four at which students begin to think proportionally. At this level students are more likely to compare results additively, e.g. “I expected six of the ten times to be a red-blue mix but only four of them were. That’s two away.” Note a proportional view is to see these four outcomes as four out of ten, $\frac{4}{10}$, a fractional idea.

It is important for students to make the connection between what happens in contrived situations, where they can calculate theoretical probabilities from models of all the outcomes, and realistic situations in which they can only estimate probabilities from sampling. Statisticians encounter these situations all the time whenever they carry out a survey or experiment, e.g. finding out the occurrence of measles per thousand people. Discuss real life situations where chance is important and ask the students how they would find the probability of a specific outcome, e.g. the chances of the school bus getting home without stopping at any lights or road signs, the chances of winning a raffle, the chance their teacher being left-handed. Some situations allow exact calculation of theoretical probability, e.g. the chance of winning the raffle form the number of tickets, but most require sampling. This raises the issue of the sample required to estimate the probability, e.g. Is asking 10 teachers enough? How many teachers would be enough? In surveys the absolutely reliable way is to sample the whole population, e.g. ask all the teachers in New Zealand, but this is usually impractical. So balancing the size and convenience of sample is always a trade off. Look for students to develop a growing understanding that big samples are more reliable than small samples in other types of statistical investigations they carry out.

In progressing towards level four students need to begin assigning numbers to chance. This involves an understanding of the part-whole nature of fractions that is less tangible and more difficult than other common models, like shapes and sets. For example, in the marbles situation one out of the ten possible outcomes is two red marbles. The chance can therefore be expressed as the fraction, $\frac{1}{10}$. A certain chance has a probability of one and impossible event has a probability of zero. In the marbles in the can situation you are certain to get at least one blue or red marble each time since ten out of ten of the possible outcomes have a red or blue marble in them, $\frac{10}{10} = 1$. It is impossible to get a white marble as none of the outcomes involve a white marble, $\frac{0}{10} = 0$.

Restrict the fractions used to those students encounter in number work, such as halves, thirds, quarters, fifths, eighths and tenths. Connect the models of possible outcomes to other part-whole representations such as area models, e.g.



Possible Resources
Figure It Out, Statistics Levels 3, Pages 18 – 24.
Figure It Out, Statistics Levels 3-4, Pages 18 – 24.
nzmaths website units:
http://www.nzmaths.co.nz/statistics/Probability/WhatsInTheBag.aspx
http://www.nzmaths.co.nz/statistics/Probability/Spinning.aspx
http://www.nzmaths.co.nz/statistics/Probability/CountingOnProbability.aspx
http://www.nzmaths.co.nz/statistics/Probability/Predictaway.aspx
http://www.nzmaths.co.nz/statistics/Probability/longrunning.aspx
Learning Objects
http://www.nzmaths.co.nz/LearningObjects/S3.aspx
Dice Duels, Random or Not, Mystery Spinner, Spinners, Foul Food Maker
http://nlvm.usu.edu/en/nav/frames_asid_305_g_3_t_5.html
http://nlvm.usu.edu/en/nav/frames_asid_310_g_3_t_5.html
http://nlvm.usu.edu/en/nav/frames_asid_186_g_3_t_5.html?open=activities
http://illuminations.nctm.org/ActivityDetail.aspx?ID=143