Supporting Multiplicative Thinking: Multi-digit Multiplication Using Array-based Materials

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This paper describes research on the classroom practices of seven teachers who taught a lesson on multi-digit multiplication to their year 7–8 students using array-based materials. The students' understanding of multi-digit multiplication just before the lesson is contrasted with their performance several weeks after the lesson (following a second lesson, on divisibility by nine). Differences among the teachers in the ways they taught the lesson are examined in relation to the students' continued understanding of multiplication. Teachers who emphasised the rectangular structure of multiplication within the context of arrays had students who made greater progress in understanding multiplication. Introduction of traditional algorithms or the "grid" method before using arrays appeared to interfere with the students' understanding of multiplication.

The Numeracy Development Projects (NDP) have now been offered to almost all primary (year 1–6) and intermediate (year 7–8) schools in New Zealand. Analysis of data on the stages of the Number Framework reached by students at the year 8 level shows that fewer than half of these students reach stage 7 (advanced multiplicative part–whole thinking), which, according to the national numeracy expectations, the majority of students at level 4 (year 8) should have reached by the end of that year (Ministry of Education, n.d.). Although the findings from the Longitudinal Study show a greater proportion of year 8 students reaching stage 7, there is evidence to suggest that the students in the Longitudinal Study are more representative of high-decile schools than typical of New Zealand schools overall (Thomas & Tagg, 2008; Young-Loveridge, 2008).

One of the major goals of today's mathematics instruction is to help students understand the structure of mathematics (Lambdin & Walcott, 2007). The greater focus on mathematics structure can be seen in the mathematics and statistics learning area of *The New Zealand Curriculum* (Ministry of Education, 2007d). In contrast to the previous curriculum (Ministry of Education, 1992), where no mention was made of changes in the nature of thinking or of problem solving over year levels, there is a clear progression in the 2007 document (shown in the achievement objectives under number strategies) from simple additive strategies with whole numbers and fractions (level 2), to additive and simple multiplicative strategies with whole numbers, fractions, decimals, and percentages (level 3), to a range of multiplicative strategies when operating on whole numbers and simple linear proportions, including ordering fractions (level 4), to reasoning with linear proportions (level 5), and to applying direct and inverse relationships with linear proportions (level 6). These progressions are closely aligned with the Number Framework, a key aspect of the NDP (see Bobis, Clarke, Clarke, et al., 2005; Ministry of Education, 2007a).

The work of Mulligan and colleagues supports the idea that students' appreciation of structure and pattern may be at the heart of differences between high and low achievers in mathematics (Mulligan & Mitchelmore, 1997; Bobis, Mulligan, & Lowrie, 2008; Mulligan, Prescott, & Mitchelmore, 2004). Their research shows that low achievers in mathematics do not appear to notice structure and regularity in mathematics, but interventions that draw their attention to structure and pattern can bring about substantial improvement in their mathematics learning.

The literature on multiplicative thinking and reasoning has been growing steadily over the past decade or so. According to Baek (1998), "understanding multiplication is central to knowing mathematics"

(p. 151). The importance of the understanding of multiplication and division for later mathematics has been affirmed in recent writing about the Curriculum Focal Points developed in the United States by the National Council of Teachers of Mathematics (NCTM) (see Beckman & Fuson, 2008; Charles & Duckett, 2008; NCTM, n.d.). Multiplicative reasoning is seen as one of three crucial mathematics themes (along with equivalence and computational fluency) that are interwoven through the Content Standards for the middle grades, forming the foundation for proportional reasoning (NCTM, 2000a, 2000b).

There are several major differences between additive and multiplicative thinking. For example, multiplication and division have proportional structure, whereas addition and subtraction have part–whole structure (Sophian, 2007). This means that multiplicative partitioning must involve equal-sized parts or groups, whereas additive partitioning often involves breaking numbers up into unequal-sized parts. Understanding multiplicative relationships depends on understanding the concept of a unit, and "that is generally developed first in the context of additive reasoning" (Sophian, 2007, p. 103). It is in considering units of quantification other than one that the need for multiplicative relations becomes clear – the unit may be a group (for example, a pair, a trio, or any other composite unit), or it may be a fractional quantity (for example, one-half, one-third, and so on). When students are young, they often don't understand the importance of keeping units constant and have a tendency when doing equal sharing to divide a continuous quantity into a particular number of pieces while ignoring the size of the pieces (Sophian, 2007).

A variety of definitions for multiplicative thinking and reasoning have appeared in the literature. According to support material for New Zealand's NDP (Ministry of Education, 2007c), multiplicative thinking involves:

> constructing and manipulating factors (the numbers being multiplied) in response to a variety of contexts ... [and] deriving [unknown results] from known facts using the properties of multiplication and division [commutative, associative, distributive, inverse]. (P. 3)

Multiplicative reasoning is far more complex than additive thinking and can involve a number of processes, such as: grouping; number-line hopping; folding and layering; branching; making grids or arrays; area, volume, and dimension; steady rise (slope); proportional reasoning; and number-line rotation (for integer multiplication). However, it has been argued that "the most flexible and robust interpretation of multiplication is based on a rectangle" (Davis, 2008, p. 88), thus reinforcing the two-dimensionality of multiplication. An area-based interpretation can be used to show how and why the algorithm for multi-digit whole-number multiplication works and can be extended to multiplication of decimal fractions, common fractions, algebraic expressions, and other continuous values (Davis, 2008; Young-Loveridge, 2005a, 2005b).

In contrast to multiplicative thinking, additive thinking is a linear process, involving a single dimension. Number-line models typically show addition and subtraction as movement either forwards (addition) or backwards (subtraction) along a line. Hence, the use of a repeated-addition strategy to solve a multiplication problem is less advanced than one involving partitioning, manipulating, and recombining quantities (see Ministry of Education, 2006). The inclusion of array diagrams as well as number-line models in the revised edition of *Book 1: the Number Framework* (Ministry of Education, 2007a) helps by showing collections-based (based on partitioning) as well as counting-based (based on skip counting or repeated addition) conceptions of the number system (Yackel, 2001) and provides richer, more flexible models of multiplication and division.

Along with the increased emphasis on multiplicative thinking have come expectations about when students should be able to use multiplicative structure. In New Zealand, there is an expectation that, by the end of year 8, students should be able to reason multiplicatively (Ministry of Education, n.d.).

However, with evidence indicating that only about 33–50% of year 8 students have good control over multiplicative structures (Young-Loveridge, 2007, 2008), it is important to understand more about the challenges for teachers at the year 7–8 level in helping their students become multiplicative thinkers.

This study explores the teaching of multi-digit multiplication using array-based materials in order to understand how different teaching approaches might impact on students' understanding of multiplication.

Method

Participants

Seven female teachers (A–G) working at the year 7–8 level (11- to 13-year-olds) from four schools (one intermediate school, one full primary school, and two middle schools) agreed to participate in the study. The decile ranking[1] of the schools in the study ranged from 2 (low) to 9 (high), reflecting the wide range of socio-economic backgrounds of the students. Teachers varied in years of teaching experience from approximately 1.5 years to 20 years. Likewise, the teachers' experience working with the NDP approach ranged from one to seven years. Each teacher chose a group of students to work with on enhancing multiplicative thinking. A combined total of 46 students took part in lessons and assessments used in the study (two other students, B1and F4, were present for only one of the lessons and are therefore not included in Table 1).

Procedure

Researchers visited each teacher twice. At the first visit, the students were given written assessment tasks to complete before the lesson, with instructions to "explain how you worked out your answer. Where possible, draw a diagram to help show your thinking." The eight tasks included: three about whole-number multiplication; two that involved deriving answers from information given and known number facts (If 4 x 30 = 120, what is 4 x 28? If 5 x 9 = 45, what is 5 x 18?); and one multi-digit multiplication problem (What is 11 x 99?). The teacher then taught a lesson on multi-digit multiplication while the researchers observed. Teachers adapted the lesson according to the knowledge of the students in the group. At some point within the lesson, the problem 23 x 37 was given to the students to solve. The teacher wore a digital audio recorder with lapel microphone to record as much as possible of the dialogue with the students. After the lesson, the researchers talked to the students, and later to the teacher, about their experiences in the lesson, in order to explore their perceptions of the lesson and any confusion that had arisen during the lesson.

After the second lesson (on divisibility by nine), the students and their teacher were interviewed again and the students were given written assessment tasks related to the two lessons. The key task for this study involved students solving the problem 23 x 37 =, showing on a dotty array (provided) how they would solve the problem, and then explaining their answer below the array. In most cases, the interval between the two lessons was between two and three weeks. All teachers taught the same two lessons taken from the support materials for the NDP on teaching multiplication and division (Book 6). This research paper focuses on the first lesson, Cross Products: Multiplication with multi-digit numbers using arrays (Ministry of Education, 2007c, pp. 67–70), and relevant written tasks given before and after the lesson.

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1 Each school in New Zealand is assigned a decile ranking between 1 (low) and 10 (high), based on the latest census information about the education and income levels of the adults living in the households of students who attend that school.

Results

Students' Prior Knowledge

In the pre-lesson tasks, most (36 out of 46) students were able to find the answer for 4 x 28, and all successfully solved 5 x 18 (see Table 1). Twenty-two students used a rounding and compensation strategy to solve the first problem (4 x 28), deriving their answer by using a combination of the information given and known number facts (for example, 4 x 30 = 120, 4 x 2 = 8, so 4 x 28 = 120 – 8 = 112). One student also used rounding and compensation for the second problem (5 x 18), starting with her knowledge of 5 x 20 = 100, then taking off 10 to get her answer of 90. The majority of students (29) used a doubling and halving strategy to solve 5 x 18 (for example, 5 x 9 = 45, so 5 x 18 = 2 x 45 = 90). Some students (seven on each problem) ignored the information given and instead used standard place value partitioning to work out their answers (4 x 20 = 80 and 4 x 8 = 32, so 80 + 32 = 112; and 5 x 10 = 50, 5 x 8 = 40, so 50 + 40 = 90). Another group (seven on the first problem and nine on the second) used the standard vertical algorithm to work out their answers. Some students made minor errors in calculating their answers to the first problem.

Some students experienced difficulty with the problem 11 x 99, with fewer than half (n = 20) of the students getting the correct answer. Of the correct answers, five students used a rounding and compensation strategy, taking 11 from 1100 to get an answer of 1089, another five students used standard place value partitioning, adding 99 to 990 to work out the answer, and ten students used the traditional vertical algorithm. Four students did not attempt the problem.

Several misconceptions were evident in the students' responses. One notable misconception was to multiply the tens digits and the ones digits but not cross-multiply tens with ones (that is, 10 x 90 = 900, 1 x 9 = 9, 900 + 9 = 909). Four of the six students in teacher E's group used a consistent but incorrect strategy (Note: teacher E was not their usual teacher for mathematics). A different misconception was revealed for the other two students in teacher E's group, both of whom gave the answer as 999. E2 wrote "allways ad *[sic]* 1 more 9", while E4 wrote "x 11 means 1 extra number for the second number that has to be itself." Four students from other groups also gave the answer as 999. C4's response of "11 x 9 = 99, 11 x 99 = 999" suggests that this student may also have been following the "add another 9" rule used by E2 and E4. Four students (A4, C2, D2, G1) attempted the problem using an appropriate strategy but made minor calculation errors on the problem.

The Lesson on Multi-digit Multiplication

There were many commonalities among the seven teachers in the ways they taught the first lesson. For example, six of the seven teachers used a modelling book to record discussions with the students and began by talking about their planned learning intentions for the lesson (see Higgins, 2006). Most discussed the nature of multi-digit numbers and associated issues around place value. Although all of the teachers used the dotty arrays (see Figure 1), some were photocopied onto paper for students to draw on with pencils or marker pens, leaving a permanent record of the processes used by the students. Other arrays were laminated, and the students used a whiteboard pen to record their working, which was erased between problems. Having the paper record to refer back to later was an advantage for both teachers and students. It was also beneficial for the researchers when they were checking back on what had happened during the lesson.

Table 1
Responses of Students on the Multiplication Problems Given Before and After the Lessons

(Correct responses are italicised in bold; RC = rounding and compensation; PVP = place value partitioning; DH = doubling and halving; Alg = algorithm; RA = repeated addition; B = border; P = partitioning; N = numbers shown; S = sum calculated; CPP = cross-product process.)

Student	GloSS	4 x 28 from 4 x 30	5 x 18 from 5 x 9	Before the lessons 11 x 99 =	After the lessons 23 x 37
A1	6	RC 116	***PVP***	***RC***	***BPNS 851***
A2	6	PVP 102	***DH***	Not attempted	BPN
A3	5	***RC***	***DH***	110 – 11 = 99	600 + 30 + 70 = 700
A4		***PVP***	***DH***	PVP 1080	***BPNS 851***
A5	6	***RC***	***PVP***	Not attempted	600 + 90 + 21 + 140 = 841
A6	6	***RC***	***DH***	1100 – 1 = 1999	BPN
A7	5	***RC***	***DH***	Not attempted	***BPNS 851***
B2	7	***RC***	***DH***	***RC***	BPNS 600 + 160 + 90 + 24 = 874
B3	6	***PVP***	***PVP***	***PVP***	***BPNS 851***
B4	7	***PVP***	***PVP***	Alg 1890	***BPNS 851***
B5	6	***PVP***	***Alg***	***Alg***	***BPNS 851***
B6	5	***RC***	***RC***	1100 – 9 = 1091	***BPNS 851***
B7	6	***Alg***	***Alg***	Alg 999	BPN
B8	5/6	***RC***	***DH***	11 x 99 = 999	BPN
C1	5	***RA***	***DH***	1 x 9 = 9,11 x 9 = 99	37 + 23 = 60
C2	6	PVP 104	***DH***	990 + 99 = 1098	600 + 21 = 621
C3	5	RC 118	***DH***	no working 999	20 x 30 = 500 + 21 = 521
C4	5	***RC***	***DH***	11 x 9 = 99,11 x 99 = 999	37 + 23 = 60
C5	5	***RC***	***DH***	Not attempted	600 + 140 + 21 + 9 = 770
C6	5	***RC***	***DH***	Alg 999	2 x 3 = 6 + 10 + 21 = 81
D1	6	***Alg***	***DH***	9 + 9 = 18 + 989 = 1089	***B Alg 851***
D2	5	***Alg***	***Alg***	Alg 1980	6 x 100 + 70 + 70 + 30 + 30 + 30 + 30 = 860
D3	5	***RC***	***DH***	***Alg***	***B Alg 851***
D4	5	***Alg***	***Alg***	***Alg***	***B 851***
D5	7	Alg 102	***Alg***	***Alg***	***BN 851***
D6	6	***Alg***	***Alg***	***Alg***	B 6 x 100 + 2 x 70 + 4 x 30 = 860
D7	5	***Alg***	***Alg***	***Alg***	B 600 + 140 + 80 + 80 = 851 guessed
E1	5	Alg 102	***Alg***	10 x 90 + 1 x 9 = 909	600 + 60 + 140 + 21 = 821
E2	6	PVP152	***DH***	11 x 9 = 99, 11 x 99 = 999 always ad *[sic]* 1 more 9	***B 851: I drew a line 23 dots down & 37 across & worked my way from there***
E3	6	***PVP***	***PVP***	10 x 90 + 1 x 9 = 909	624 because there are 6 full boxes & 2 7s & 4 3s = 624
E4	6	***RC***	***DH***	x 11 means 1 extra number for the 2nd number that has to be itself = 999	111 + 6 x 100 + 70 + 70 = 857
E5	6	***RC***	***PVP***	10 x 90 + 1 x 9 = 909	***600 + 140 = 740 + 90 = 830 + 21 = 851***
E6	6	RC 102	***DH***	10 x 90 + 9 = 909	20 x 30 = 600 + 7 x 3 = 621

Student	GloSS	4 x 28 from 4 x 30	5 x 18 from 5 x 9	Before the lessons 11 x 99 =	After the lessons 23 x 37
F1	8	***PVP***	***PVP***	***PVP***	***B 600 + 21 + 90 + 140 = 851***
F2	8	***RC***	***DH***	***Alg***	***BPNS 600 + 140 + 90 + 21 = 851***
F3	7	***Alg***	***Alg***	Alg = 289 (99 + 190)	***BPNS 600 + 140 + 90 + 21 = 851 Grid***
F5	8	***RC***	***DH***	***Alg***	***BP 600 + 140 + 90 + 21 = 851 Grid***
G1		***RC***	***DH***	RC 1981	BPNS 600 + 90 + 140 + 21 = 751
G2	6	***RC***	***DH***	***Alg***	***BPNS 851***
G3	8	***RC***	***DH***	***RC***	***BPNS Grid 851***
G4	7	***RC***	***DH***	***Alg***	***BPNS 851***
G5	7	RC 118	***DH***	***RC***	***BPNS Grid 851***
G6	8	***RC***	***DH***	***RC***	***BPNS Grid 851***
G7	6	RC 132	***DH***	***PVP***	***BN 851***
G8		***PVP***	***DH***	***PVP***	***BPNS 600 + 140 + 111 = 851***
G9	6	***RC***	***DH***	***PVP***	***BPNS Grid 851***

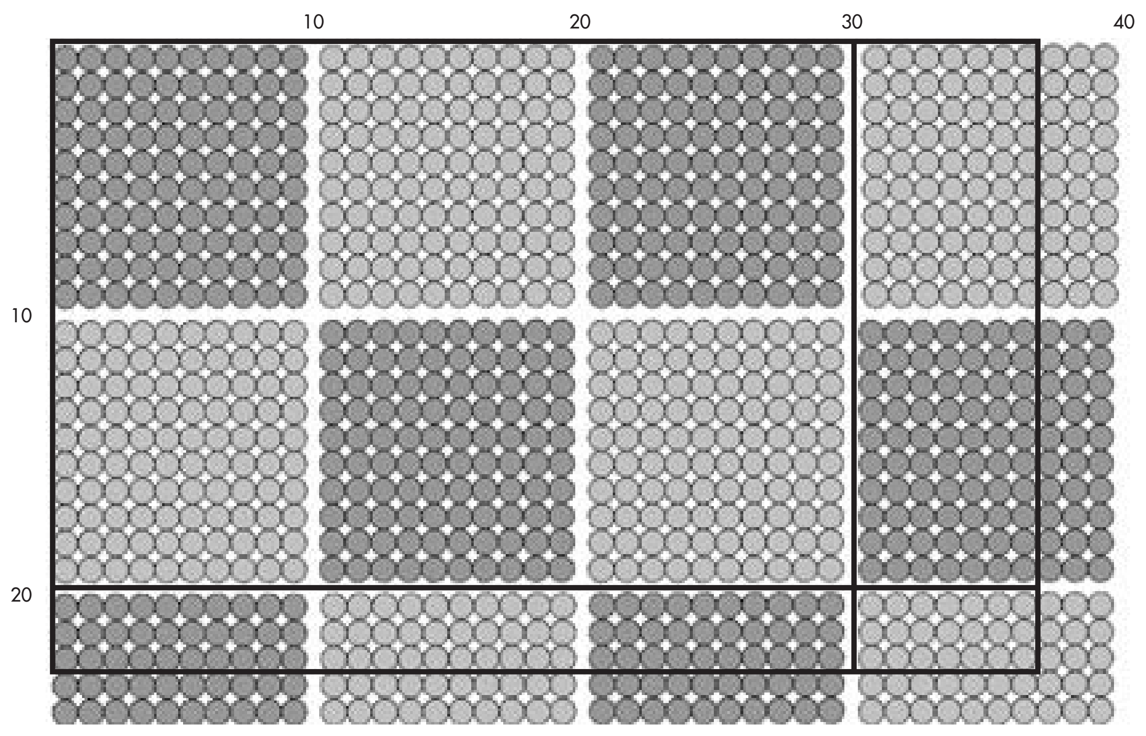

Figure 1. Example of an array showing **23 x 37** as 20 x 30 = 600; 20 x 7 = 140; 3 x 30 = 90; 3 x 7 = 21; 600 + 140 + 90 + 21 = 851

Several teachers began by introducing arrays using single-digit multiplication (for example, 5 x 6), and this appeared to be very helpful for scaffolding the idea of representing multiplication as a rectangle with sides corresponding to each of the factors. Drawing a border around the rectangle formed by the two factors turned out to be important for students' understanding. Although there were some commonalities between the seven lessons, each was very different from all the others.

Students' Understanding of Multiplication After the Lesson

The students in teacher A's group (only one of whom had answered 11 x 99 correctly before the lesson) seemed to have made the greatest progress towards understanding multiplication over the two weeks following the first lesson. Six of the seven in the group successfully used the dotty arrays to work out the partial products for 23 x 37. Three of these students added the partial products to get 851 (A5 miscalculated and got 841). Two students (A2 and A6) did not take the final step of adding the partial products. The only student who did not appear to benefit from the array materials (A3) got partial products of 600, 30, and 70 and added up to get 700. She wrote 3 x 10 = 30 (instead of 3 x 30 = 90), only noticed one group of 70, and completely overlooked the 3 x 7. It was interesting to note that for 11 x 99 before the lesson, this student had written 110 – 11 = 99, failing to notice that her answer was the same as one of the factors. Her strategies on both of these problems may be indicative of carelessness rather than misconceptions about multiplication.

Teacher A started the lesson (after first discussing the learning intention and what multi-digit numbers are) by asking the students what they noticed about the dotty array, drawing their attention to the regular structure of rows and columns of dots and the separation between each group of ten dots so that the 10 by 10 blocks of 100 dots were easy to see. She then asked the students to explain how they would show 6 x 5 = 30 on the array, eventually instructing them to draw a border around it. In the interview, teacher A commented on the importance of drawing a border around the part of the dotty array that represented the problem: "The border is definitely the key word for me – that's why I underline it [on the whiteboard]."

Teacher A made a point of discussing the meaning of key mathematics terms such as "represents" and "partitioning". She asked the students to work in pairs and then to explain each other's work to the group to ensure that they all understood what they needed to do in working with the arrays. When one student showed partitioning without putting a border around the whole problem, teacher A asked the whole group to think about what that student needed to do differently. There was a lot of discussion about partitioning, with students being asked "Has everybody got a partition?" and "Where did you put your partition?" as well as being questioned about "splitting it up".

Teacher A also made sure that her students "numbered the edge" of the array so that the connection between the factors and the array was made clear. She also talked about "factor times factor equals product", linking it to the name of the lesson (Cross Products).

Teacher A did not move to a two-digit by two-digit multiplication problem until more than halfway through the 50-minute lesson. After the students had worked through each of the partial products, teacher A got them to draw up a grid (in preparation for doing the "grid method") in their books (see Figure 2). She then constructed a model of the "cross-product process" by writing the place value partitioned factors on four separate cards (for example, 20 and 3, 30 and 7) and used string stretched between the cards to show the horizontal and diagonal connections between the four parts that were multiplied together to produce the four cross products (see Figure 3).

x	**30**	7	
20	600	140	740
3	90	21	111
	690	161	851

Figure 2. The "grid method" for calculating the answer to the problem: 23 x 37
(a variation on that shown in Ministry of Education, 2007c, p. 68)

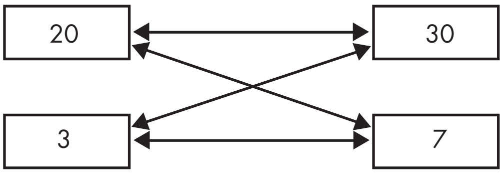

Figure 3. Model showing the "cross-product" process (Ministry of Education, 2007c, p. 69)

In her reflections on the lesson, teacher A rated the students' understanding of double-digit multiplication as "possibly a three ... on a scale of one to ten". In hindsight, it would have been valuable to ascertain how much additional work on multi-digit multiplication teacher A did with the students in further lessons before the researchers' second visit.

Teacher B's students were slightly stronger mathematically than teacher A's. Two of the seven students (B2 and B4) had been assessed as already being at stage 7 (advanced multiplicative part–whole). However, only B2 and B3 used part–whole strategies to solve 11 x 99 before the lesson.

Two weeks after the lesson using dotty arrays, all but one of teacher B's students were able to use dotty arrays to work out 23 x 37, although two students (B7 and B8) did not add their partial products together at the end to find the sum. The only student who did not succeed (B2) had accidentally drawn his array as 23 x 38, resulting in partial products of 160 and 24 instead of 140 and 21. Student B6's response was typical of students who used the dotty array with understanding to solve the problem (see Figure 4).

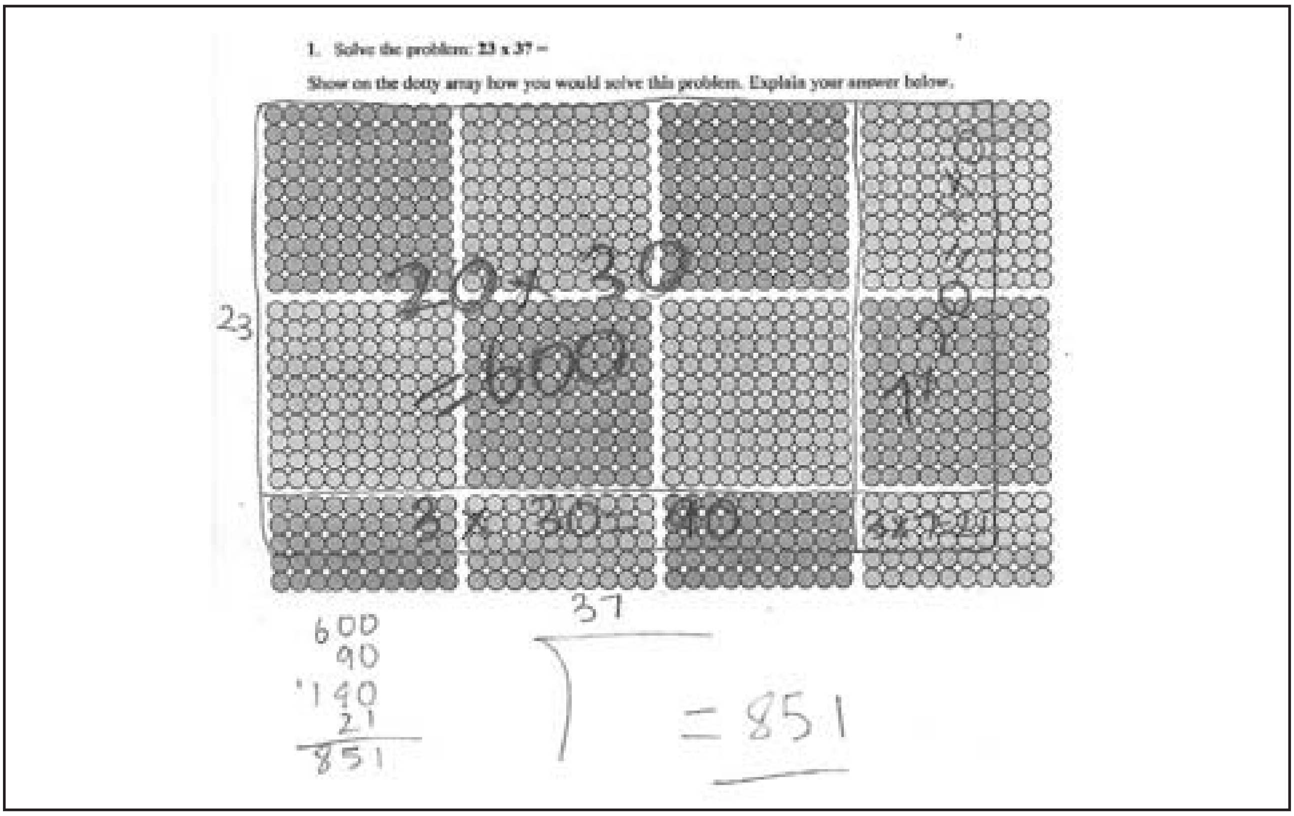

Figure 4. Student B6's record of her solution to 23 x 37 =

In her reflections on the lesson, teacher B, who taught the lesson for the first time on the day of the researchers' first visit, commented that in the future, she would start by stressing the importance of drawing a border around the whole problem. Several of her students progressed from having difficulties with 11 x 99 before the lesson to successfully working out 23 x 37 several weeks after the lesson.

In hindsight, it would have been useful to know what further work teacher B had done with her group on multi-digit multiplication between the two visits. Another interesting area to explore would have been the possible influence of teacher A on teacher B's classroom practice. Given teacher A's many years of experience as a teacher and of working with the NDP approach, together with her leadership role within the school, it seems likely that teacher B benefited from working collaboratively with teacher A.

The Impact of Traditional Algorithms

Teacher D's students were strongly algorithmic in their responses to all of the tasks given. Only one of the seven students derived 4 x 28 from 4 x 30, and two derived 5 x 18 from 5 x 9. Five of the seven used the vertical written algorithm to work out 11 x 99. Although all seven students drew a border around the dots corresponding to the problem 23 x 37, there was little, if any, partitioning (D6 was the exception, partitioning two blocks of 100 and one block of 70 within the array). Two students (D1 and D3) wrote out the vertical written algorithm as part of their explanations. It seems likely that others in the group also used the algorithm mentally to find their answers. In later discussions with teacher D, it became clear that she had focused exclusively on traditional written algorithms in her teaching of multiplication and division. She had undertaken NDP professional development quite recently but, being a second-year teacher, was still finding her feet.

Teacher E appeared to have done considerable work with her students to correct their misconceptions about multi-digit multiplication. However, the students still tended to use the traditional algorithm to work out their answers before drawing a border around the dots representing the problem. Two students (E2 and E5) found the correct answer for 23 x 37 but showed no evidence of partitioning the array – they both simply drew a border around the dots representing the problem. Student E2 wrote "I drew a line 23 dots down and 37 across and worked my way from there." Student E5 wrote "600 + 140 = 740 + 90 = 830 + 21 = 851" underneath her array. Inside her array, she appeared to have numbered the blocks of 100 with "10, 20, 30, 40, 50, 60" and then beneath, numbered across the three rows of 30 and the 3 by 7 section: "21, 42, 63, 84". It is possible that this student worked out the correct answer by applying the traditional algorithm mentally after numbering the array. Student E1 was the only student in this group to partition the array, but she miscalculated 30 x 3 as 60 instead of 90, giving an answer of 821 instead of 851.

Teacher F's students also calculated their answers before drawing borders around unconnected partial products. All four students had been assessed as being at stage 7 or 8 on the Number Framework before the lesson. It became clear during the interview that teacher F did not think that dotty arrays could help her students understand multi-digit multiplication and thought that the vertical written algorithm (which all her students applied correctly) or the grid method (partitioning each factor into tens and ones, then multiplying horizontally and diagonally to create cross products) were the best ways to solve multi-digit multiplication problems. She commented that she didn't think the array "added any value to what they [already] had".

Matching the Lesson to the Learning Needs of the Group

The responses of students in teacher C's group to the 23 x 37 problem revealed considerable confusion about the use of dotty arrays to solve multi-digit multiplication problems, suggesting that perhaps the cross-product lesson was not a good match for the learning needs of the group this teacher had chosen to work with for this lesson. None of the students were able to work out the answer to 23 x 37 two weeks after the lesson. It appeared that most of the students had major issues with place value, confusing tens with hundreds and multiplication with addition. For example, C1 coloured in three blocks of 100 and a row of seven dots (for 37) and, below this, coloured two blocks of 100 and a row of three dots (for 23), then "plused *[sic]* them together" writing the answer as 60 (the sum rather than the product of 23 and 37) (see Figure 5).

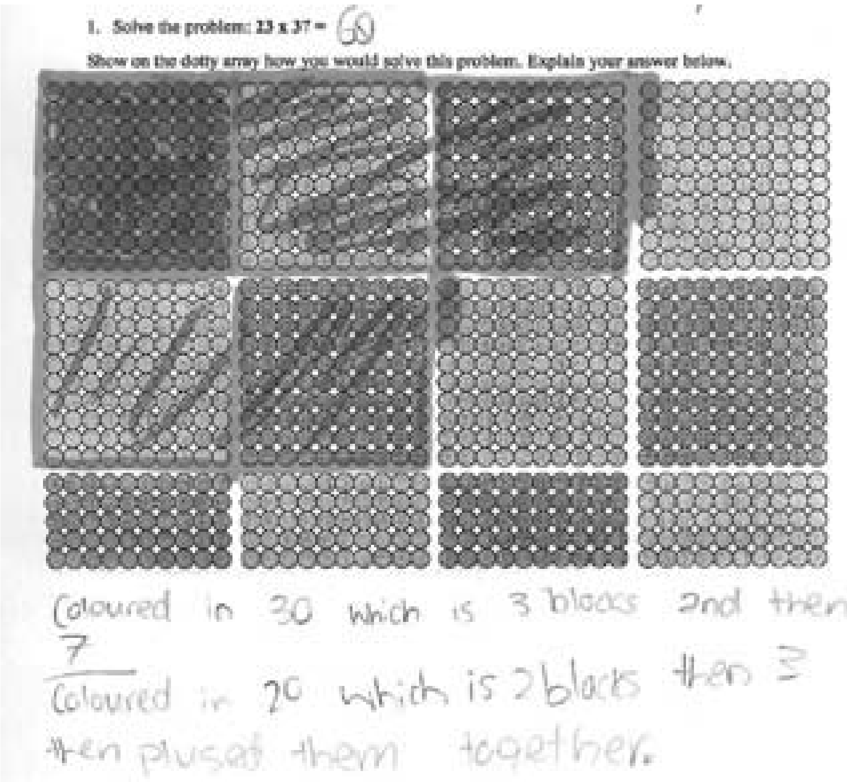

Figure 5. The response of student C1, showing confusion between tens and hundreds and between addition and multiplication

Although C4 drew a border around 37 by 23, like C1, she used a highlighter pen to colour in three blocks of 100 and a row of seven dots (for 37) and, below this, coloured two blocks of 100 and a row of three dots (for 23), giving the answer as 60. C2 drew a border around six blocks of 100 (3 across and 2 down), then a separate border around 21 dots in a 3 x 7 array to the right of the block of 600. He added 600 and 21 to get an answer of 621 (see Figure 6).

C3 drew two almost identical arrays (of 600 and 21) to C2 but cut out one block of 100, writing "20 x 30 = 500", then added the 500 and the 21 from the 3 x 7 array to get an answer of 521. C5 drew a border around an array of 30 by 20 and a border around the 7 x 20 array adjacent to the 600. However, there were also borders around 3 x 7, directly below the 30 x 20, and around a 3 x 3 array (instead of 3 x 30) below the 7 x 20 array, resulting in the sum of cross products being 770.

C6 drew a border around three blocks of 100 and highlighted a column of three dots on the top left corner of the third block (for 23). Underneath, she drew a border around four blocks of 100, highlighting a column of seven dots on the top left corner of the fourth block (for 37). She wrote: "Split the numbers, go back 2 x 3 = 6 and 3 x 7 = 21 then add 10 back onto [drawing an arrow pointing to the 3 in 2 x 3], add them together to equal 81." Her reference to "add 10 back onto" the 3 suggests she was using an "add zero" rule to adjust for the fact that the 2 was really 20 and the 3 was really 30, but she did not recognise that she needed two more zeros (as in 600), not one (as in 60). Evidence of the "add zero" rule was found on some of the work that C2 did during the lesson, where he wrote "37 x 23 = 2 x 3 add 00 = 600". It was interesting to note that before the lesson, C3, C4, and C6 had all given 999 as the answer to 11 x 99 in the initial assessment. Only C2 had correctly partitioned the 99 into 90 and 9 and calculated partial products of 990 and 99 (C2 made a slight error in summing the partial products, getting a final answer of 1098).

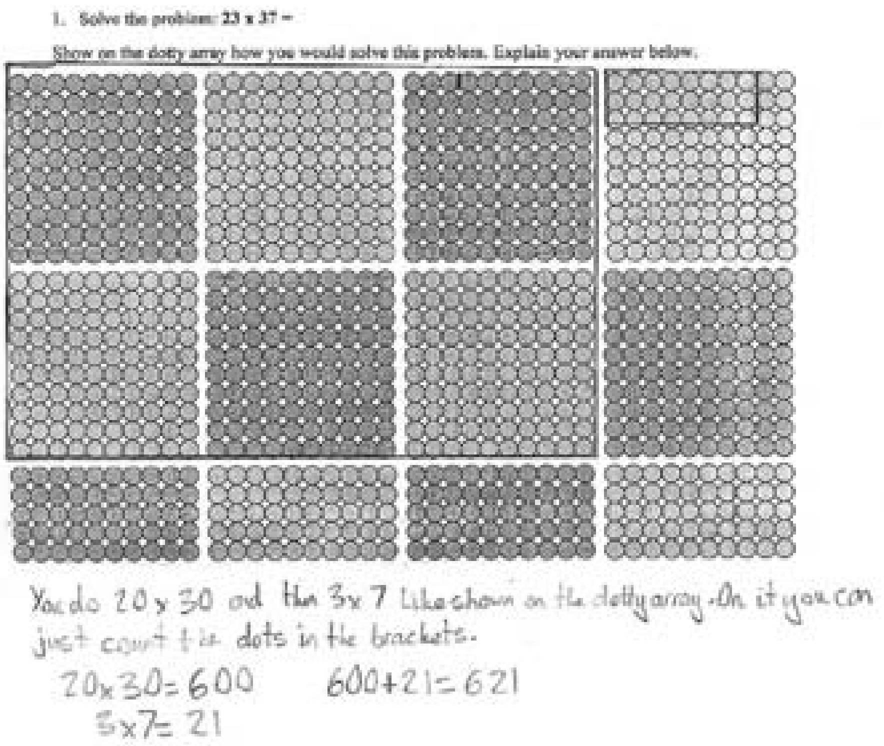

Figure 6. Student C2's response to 23 x 37 =

One reason for the confusion experienced by teacher C's students might have been that this teacher started straight into multi-digit multiplication problems rather than introducing arrays initially with single-digit problems. All but one of the students in this group had been assessed as being at stage 5 early additive part–whole thinking (C2 was assessed as being at stage 6). These students may well have been still trying to get to grips with the complexities of advanced additive part–whole thinking and might not have been ready to deal with the added challenge of working with multi-digit multiplication. It seems likely that they needed a gradual introduction to the use of arrays for single-digit multiplication before moving to work with larger numbers.

Teacher C did not emphasise the importance of first drawing a border around the whole problem. Several of her students tried to work out the partial products first and then put borders around those products, often drawing borders around separate and unconnected cross products. The result was a separate rectangle for each cross product, without coherence or connection to the original factors or the total product. The fact that teacher C accepted her students' strategy of drawing borders around unconnected partial products during the lesson suggested that this teacher did not understand how a rectangular structure can be used to represent multiplication and hence the need for a border around the whole problem before partitioning. Both the teacher and her students seemed confused and a little frustrated during the lesson. Teacher C was aware that her students had found the lesson difficult, later saying "They found it really, really hard."

Teacher C's comment that her students "still struggle a little bit with place value" showed her awareness that this probably contributed to their difficulty using the arrays. Teacher C had seemed somewhat apprehensive about her lesson being observed and audio-recorded, and this may have also played a part in her confusion during the lesson. This was confirmed in the second interview when teacher C commented that she "felt a lot more at ease" during the second lesson. In hindsight, it would have been good to explore with teacher C her thoughts on possible reasons for her students' difficulties with the 23 x 37 problem.

The lesson with dotty arrays may not have been quite such a good match for the group that teacher G chose to work with because they were already quite strong multiplicative thinkers. They had been assessed as already being mostly at stage 7 or above (five out of eight) and started the lesson with a good understanding of multiplication (8 out of 9 had been able to solve 11 x 99 before the lesson). The students tended to use the grid method to work out their answers to 23 x 37. However, all but one student (G7) showed appropriate partitioning inside the border they had drawn around the dots in the array representing the problem. The only student who did not calculate the correct answer (G1) had made a slight calculation error at the last step when adding up the partial products. It was interesting to note that student G1 had also made a slight computation error on 11 x 99, when the student tried to subtract 11 from 1100 and got an answer of 1981. Algorithms were used by only two students (G2 and G4), and then only to solve 11 x 99.

The responses of teacher G's students reflected a strong emphasis on understanding. At the time of the research, teacher G was nearing the end of her first year of NDP professional development and had been focusing on multiplication and division with her class. Although the group she chose to work with almost certainly benefited from the work with the arrays in consolidating their understanding of multi-digit multiplication, it would have been interesting to have observed a group who began the lesson with less prior knowledge and understanding of multiplication.

Teacher Reflections and Comments

The transcription of the teachers' language during the lesson was very useful in helping to identify differences between their approaches that may have been crucial in explaining why some students learned more effectively than others. Teacher A's approach was notable for the way in which she moved very slowly from single-digit multiplication to two-digit multiplication, starting with one-digit by one-digit problems, then moving to one-digit by two-digit problems, before introducing two-digit by two-digit problems. This contrasts with the approach of some other teachers (for example, teacher C) who began the lesson with the 23 x 37 problem. Teacher A also made a particular point of focusing on mathematics language, checking with her students about their understanding of key terms at regular intervals. As well as stressing the importance of the border, she also talked a lot about partitioning, repeatedly asking her students how they had split the numbers up or "made a partition". Teacher A also made a point of getting the students to put the numbers on the sides of the rectangle ("numbering the edge") and to draw a line between each of the digits to highlight the cross products.

Teacher A differed from the other teachers in that she had previously taught the lesson on cross products to another group of students and had clearly refined her technique in response to her previous experience. All of the teachers commented on using the grid method, in which each multi-digit number is broken down according to place value and each part is multiplied with all/both of the parts in the other factor. However, for some students, it seemed to have become just like the traditional written algorithm – a mindless procedure executed without understanding. Only teacher A seemed to appreciate from the outset how the array model enabled students to get a picture of the magnitude of the quantities in the partial products. She was clear about the value of using the array initially to build understanding of the multiplication process. She then followed this with the grid method and the cross-product process (using cards for the place value partitioned factors and string stretched between the parts of one partitioned factor and the parts of the other factor). The formal written algorithm was introduced last as just another way of solving the problem.

It would also have been good to see the lesson go beyond using the dotty arrays on to hand drawing rectangles to represent the problems solved using the dotty arrays.

Discussion

The findings of this study indicate that arrays can be useful for enhancing students' understanding of multi-digit multiplication. Teachers' use of dotty arrays to represent multi-digit multiplication as a rectangle with sides corresponding to the two factors was associated with improved performance on multiplication problems a few weeks after the lesson.

The findings are consistent with the view that "the most flexible and robust interpretation of multiplication is based on a rectangle" (Davis, 2008, p. 88). An advantage of dotty arrays is that they help students to appreciate differences in the magnitude of partial products and the impact of place value on the size of the sections (that is, partial products) within an array. For example, the six blocks of 100 dots representing 20 x 30 (the "tens") was substantially larger than the 21 dots in the 3 x 7 (the "ones") array.

The results reported here support the views of Mulligan and colleagues (Mulligan & Mitchelmore, 1997; Bobis et al., 2008; Mulligan et al., 2004) that coming to understand the underlying structure of the mathematics is vitally important for effective mathematics learning. Some students who were only using vertical written algorithms or the grid method appeared to have difficulty understanding how arrays could be useful for solving multiplication problems. An emphasis on procedural knowledge and rules, as reflected in the use of algorithmic approaches to multiplication, may undermine conceptual understanding. As Pesek and Kirshner (2000) have shown, once students have been taught to use standard written algorithms, it can be extremely difficult to then try to help them develop relational understanding.

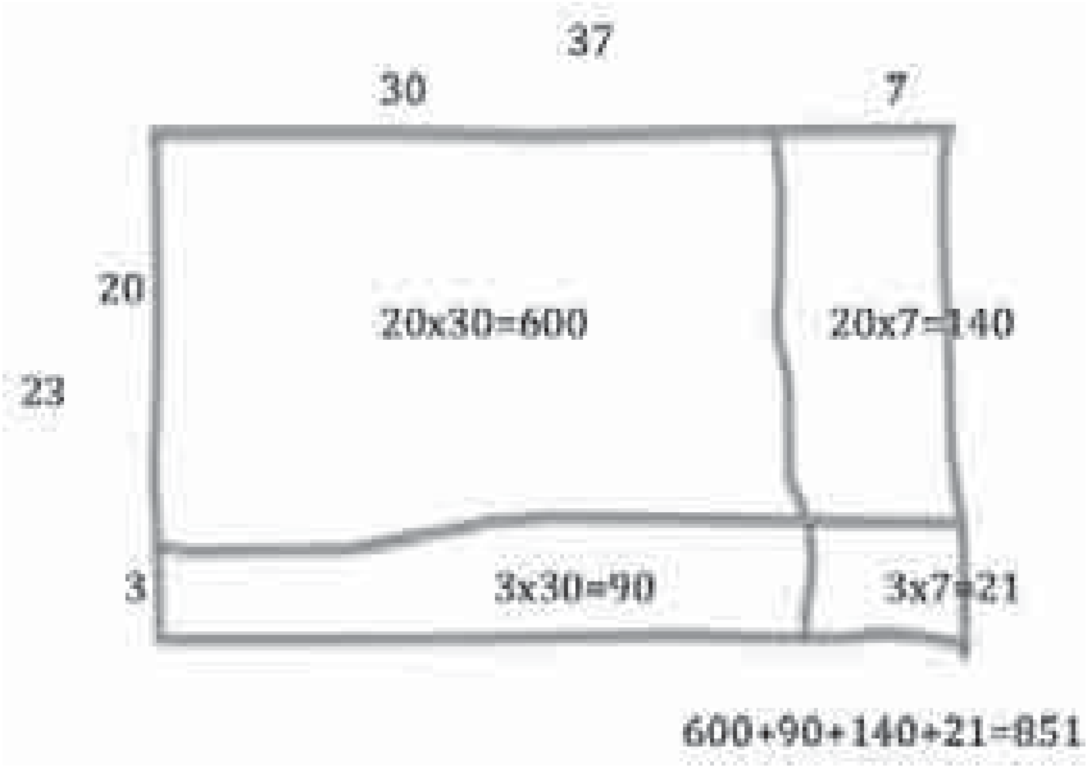

Figure 7. An example of a hand-drawn rectangle showing an array-based solution strategy for the problem: 23 x 37

The researchers have concluded that it is rather unfortunate that in the NDP support materials (Book 6), the lesson on traditional written algorithms for multiplication (entitled Paper Power) comes before rather than after the lesson on using dotty arrays to make sense of multi-digit multiplication. The order of the lessons in the planning sheets also has the traditional written algorithms for multiplication coming before instead of after the lesson on dotty arrays (see www.nzmaths.co.nz). The introduction of hand-drawn rectangles as a means of representing the array would provide a useful intermediary step between the dotty arrays and the grid method (see Figure 7). An example of hand-drawn arrays in Book 6 would show teachers that pictures and diagrams can provide a useful means of recording the problem solving in multi-digit multiplication. The digital learning objects on the nzmaths website allow students to play around with different ways of partitioning arrays to make multiplication easier and to reinforce the two-dimensional structure of multiplication problems (see The Multiplier, nzmaths, n.d.).

It is interesting to note that none of the teachers referred to the distributive property, either in their work with the students or in their discussions with the researchers. It is the use of the distributive property (whether or not students refer to it by name) that distinguishes those who are advanced multiplicative thinkers and can therefore construct and manipulate factors in response to a variety of contexts and derive the answers to unknown problems from known facts, using the properties of multiplication and division (that is, commutative, associative, distributive, inverse) (Ministry of Education, 2007c). It would be good to see further information about these number properties, particularly in Book 3, where the teaching model is described (Ministry of Education, 2007b).

Some teachers may not be aware that the number properties that are part of the teaching model are the same number properties that they may have been taught about when they were at school in the so-called "New Maths" era. Understanding how numbers can be added or multiplied in any order (the commutative property) or grouped in any way (the associative property, as reflected in halving and doubling, tripling and "thirding", or quadrupling and quartering) underpins the part–whole strategies that are characteristic of students' thinking at stages 5 and above on the additive and multiplicative domains. Unique to multiplication is the distributive property, which allows factors to be partitioned additively so that students can derive answers to unknown multiplication problems by partitioning factors in such a way that known facts can be used to solve parts of the problem (that is, partial products) and the parts can then be joined together to create the final product, for example, $23 \times 37 = (20 + 3) \times (30 + 7) = (20 \times 30) + (20 \times 7) + (3 \times 30) + (3 \times 7) = 600 + 140 + 90 + 21 = 851$.

If we are to support teachers in helping their students to become advanced multiplicative, then it is important to provide the teachers with the principles that underpin the processes we are encouraging them to use. While number properties (commutative, distributive, associative, inverse) are mentioned in Book 6, examples given, and suggested vocabulary included, no definitions are provided to help teachers appreciate that the essence of the associative property is about changing the *grouping* and the essence of the distributive property is about *partitioning* factors into convenient chunks. It is likely that many teachers do realise that the commutative property is about changing the *order* of factors ("turn-abouts"), but it would be interesting to know how many of them have also made links with the number properties mentioned in the teaching model.

The inverse property refers to reversing and doing and undoing, but it should also show how multiplication of the product by a unit fraction that is the reciprocal of one factor results in the other factor, in just the same way that division reverses multiplication (for example, $7 \times 4 = 28$, so $28 \div 4 = 7$, and $28 \div 7 = 4$; also $\frac{1}{4}$ of $28 = 7$, and $\frac{1}{7}$ of $28 = 4$).

Book 1 can also be used to make more links with the number properties now that the revised version shows how array models can be used to model multiplication as well as number-line models (Ministry of Education, 2007a).

The findings of this study have some important implications for the grid method that seems to have become a popular method for teaching multiplication (see Figure 2). The results indicate that the grid method is possibly being taught as an alternative to the traditional written algorithm and may suffer from the same problem of being applied in a mindless way (as rules and procedures without meaning) that has contributed to warnings and cautions for the vertical written algorithm.

As with the related cross-product process (see Figure 3), the understanding developed when students draw arrows (both horizontal and diagonal) between the place value partitioned factors and then add up the partial products that result from these multiplications is not much greater than the understanding they develop from using traditional vertical written algorithms.

What is important for students to appreciate is that, when the two "tens parts" are multiplied together, the partial product is extremely large, whereas the multiplication of the two "ones parts" yields a relatively small partial product by comparison. They need to see where the "tens" by "ones" and the "ones" by "tens" partial products come from in the rectangular structure that represents the problem as a whole (see Davis, 2008). It was very striking to see the number of students in teacher C's group whose answer to 23 x 37 was less than 100, even though they had apparently done quite a bit of work with the grid method (see Figure 2).

The differences between teacher A's approach and the approaches of the other teachers highlights the importance of mathematics language and the need for teachers to help their students become familiar with the terms used to describe the processes that are involved in solving problems by applying their understanding of number properties and using part–whole strategies. The challenges for teachers of coming to understand that language themselves and developing suitable ways for helping their students learn about it have probably been underestimated.

The findings of this study are consistent with the idea that teachers' knowledge and understanding of mathematics has a great impact on their teaching (see Ball, Hill, & Bass, 2005; Hill, Rowan, & Ball, 2005; Hill, Schilling & Ball, 2004). It is clear that multiplicative reasoning is complex and multifaceted. There are many challenges for teachers in fully understanding the many aspects of multiplicative thinking so that they can then decide on the best ways to support their students in acquiring that conceptual understanding. The findings of this study suggest that teachers need considerable support in coming to understand what multiplicative thinking involves and how they can use NDP resources, such as the booklets (for example, Ministry of Education, 2007a, b, c), to help their students. For teachers working at the upper primary level, the two years of professional development provided by the NDP was not enough for most of them to become familiar with superficial aspects of the Number Framework and the assessment tools, let alone build an in-depth understanding of the NDP approach overall.

The two years' of professional development that most of these intermediate (year 7–8) teachers had been given does not appear to have been sufficient for them to fully understand the complexities of multiplicative thinking. Teacher A, on the other hand, had more than 20 years of teaching experience, including four years' experience with the NDP, as well as considerable ongoing professional development provided because of her role as a lead teacher in her school and as a numeracy coach for teachers in other schools.

The NDP has provided teachers with a great start on the journey but, on average, the distance travelled so far is small compared with the distance to the final destination. This destination is the point at which all students will be taught by teachers who have the necessary levels of content knowledge and pedagogical content knowledge in mathematics to ensure that the majority of their students do reach the curriculum level expectations.

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